On the isomorphism problem for virtually free groups

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[1]Based on joint work with Géraud Sénizergues and Volker Diekert

Outline

- Introduction
- Main result: complexity of the isomorphism problem for virtually free groups

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- Cuts and structure trees

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Word problem for a group $G = \langle \Sigma \mid R \rangle$: Given a word $w \in \Sigma^*$, is $w =_G 1$?

$$\mathrm{WP}(G) = \{\, w \in \Sigma^* \mid w =_G 1 \,\}$$

Warm-Up: Isomorphism problem for free groups

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- Add relations $ab = ba$ for $a, b \in \Sigma$ and $a^2 = 1$ for $a \in \Sigma$.
- Same for $\langle \Sigma' \mid R' \rangle$.
- Use linear algebra to check isomorphism of $\mathbb{F}_2$ vector spaces.

Context-free Languages

A context-free grammar is a tuple (V, Σ, P, S) with $P \subseteq V \times (V \cup \Sigma)^*$:

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$\Sigma = \{:=, ;, \textbf{if}, \textbf{then}, \textbf{else}, \textbf{endif}, \textbf{while}, \textbf{do}, \textbf{endwhile}, (,), +, *, =, \neg, \wedge\}$,
$V = \{ A, B, C, X \}$,

$C \to X := A \mid C; C \mid \textbf{if } B \textbf{ then } C \textbf{ else } C \textbf{ endif} \mid \textbf{while } B \textbf{ do } C \textbf{ endwhile}$

$A \to X \mid (A + A) \mid (A * A)$

$B \to A = A \mid \neg B \mid B \wedge B$

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$$S \to aS\overline{a}S \mid \overline{a}SaS \mid bS\overline{b}S \mid \overline{b}SbS \mid 1$$

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Example

$K = \mathrm{WP}(F_2)$

$L =$ freely reduced words $\qquad \rightsquigarrow K \cap L = \{1\}$ is context-free

Virtually free groups

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Virtually free presentation:

- basis X of F,
- a system of representatives $R \subseteq G$ of $F \backslash G$
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$$qa = fr.$$

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Example

Let $F = \mathbb{Z} = \langle x \rangle$, $Q = \mathbb{Z}/2\mathbb{Z}$, $R = \{1, a\}$

with rules $ax = xa$, $aa = 1$.

The isomorphism problem for virtually free groups

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$$F_3 = \mathbb{Z} = \langle z \rangle\,, \qquad Q_3 = \mathbb{Z}/2\mathbb{Z}, \qquad R_3 = \{\,1, c\,\}$$

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Then $G_1 \cong G_3 \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (via $z \mapsto x$, $c \mapsto ax$) and $G_2 \cong \mathbb{Z}$.

The isomorphism problem for virtually free groups

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Theorem (Sénizergues, W. 2018)

The isomorphism problem

- *with virtually free presentations as input is in* PSPACE,
- *with context-free grammasr as input it is in* $\mathsf{SPACE}(2^{2^{\mathcal{O}(n)}})$.

Definition (Graph of Groups)

A graph of groups $\mathcal{G}$ is a connected graph $Y = (V(Y), E(Y))$ and

1. for each vertex $P \in V(Y)$, a vertex group G_P,
2. for each edge $y \in E(Y)$, an edge group $G_y \leq G_{s(y)}$.
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$$\pi_1(\mathcal{G}, T) = F(E(Y))$$

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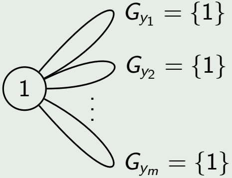
Example

F_m



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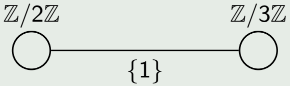

$$\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

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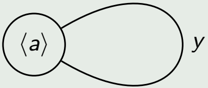

$\mathrm{BS}_{p,q} = \langle a, y \mid ya^p y^{-1} = a^q \rangle$ edge groups $G_y = \langle a^p \rangle$ and $G_{\overline{y}} = \langle a^q \rangle$ and isomorphism $a^p \mapsto a^q$

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Theorem (Guirardel, Levitt 07; Clay, Forester 09)

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be reduced finite graph of groups with finite vertex groups. Then $\pi_1(\mathcal{G}_1, T_1) \cong \pi_1(\mathcal{G}_2, T_2)$ iff $\mathcal{G}_1$ can be transformed into $\mathcal{G}_2$ by a sequence of slide moves.

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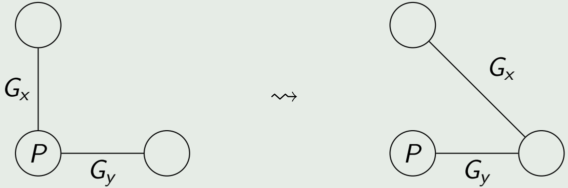
Example (Slide move)



If there is some $g \in G_P$ such that $g^{-1} G_x^x g \leq G_y^y$.

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Corollary

It can be decided in NSPACE(n) *whether $\pi_1(\mathcal{G}_1, T_1) \cong \pi_1(\mathcal{G}_2, T_2)$ given two graph of groups $\mathcal{G}_1$ and $\mathcal{G}_2$ with finite vertex groups.*

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Krstić's proof.

- For both input groups guess a GoG + an isomorphism
- verify that the guesses are correct
- check the two GoGs for isomorphism □

New approach

- Guess a GoG + an isomorphism.
- Check that the guess is correct.
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Show that the GoG and the isomorphism are “small”.

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Input: a c.f grammar for $\mathrm{WP}(G)$*,*
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Theorem (Sénizergues, W. 2018)

The following problem is in NP*:*
Input: a group G *given as virtually free presentation,*
Compute a GoG $\mathcal{G}$ *with finite vertex groups and* $\pi_1(\mathcal{G}, T) \cong G$*.*

Small graph of groups

Main Lemma

Let G be given as context-free grammar of size $N \geq 4$ for $\mathrm{WP}(G)$. There is a graph of groups $\mathcal{G}$ over Y and an isomorphism $\varphi : \pi_1(\mathcal{G}, T) \to G$ with

1. $|V(Y)| \leq N^{50 \cdot 2^N}$,
2. $|G_P| \leq N^{50 \cdot 2^N}$ for all $P \in V(Y)$,
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If G is given as virtually free presentation of size $M \geq 4$, then

1. $|V(Y)| \leq M + 1$,
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Small graph of groups

Main Lemma

Let G be given as context-free grammar of size $N \geq 4$ for $\mathrm{WP}(G)$. There is a graph of groups $\mathcal{G}$ over Y and an isomorphism $\varphi : \pi_1(\mathcal{G}, T) \to G$ with

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Proof of the Main Lemma

Theorem (Sénizergues 96)

Let G be a context-free group and N be the size of a c.f. grammar in Chomsky normal form for its word problem. Then

- $|H| \leq N^{12 \cdot 2^N + 10}$ *for every finite subgroup* $H \leq G$,
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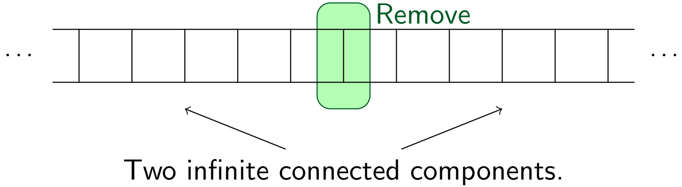
$\rightsquigarrow$ Rest of the talk

Muller and Schupp's Proof (1983)

- Every infinite virtually free group has more than one end.

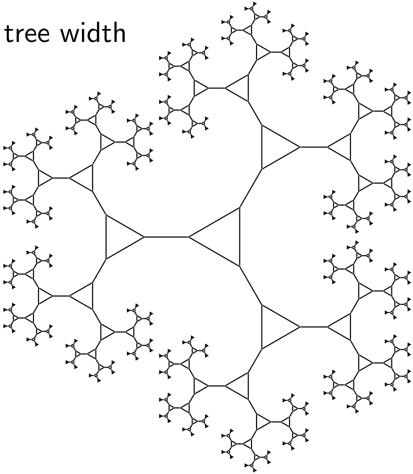
  Example: $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



- Stallings' Structure Theorem: every group with more than one end splits as HNN extension or amalgamated product over a finite subgroup.
- G finitely presented $\rightsquigarrow$ G is accessible: this splitting happens only finitely many times (Dunwoody 1985).

Let $\Gamma(G)$ be the Cayley graph of a context-free group G. Then:

- $\Gamma(G)$ is quasi-isometric to a tree
- $\Gamma(G)$ has finite tree width



The Cayley graph of $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ has finite tree-width.

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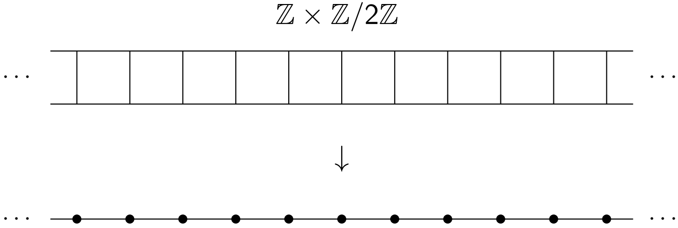

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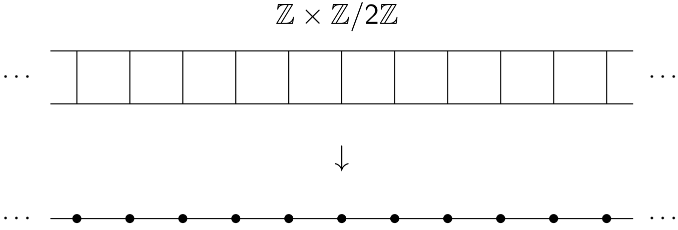

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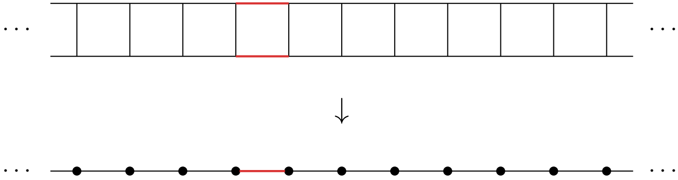

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Cuts

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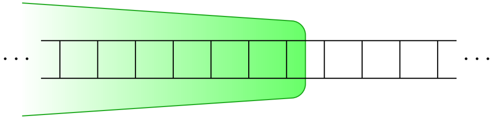
-  ✓

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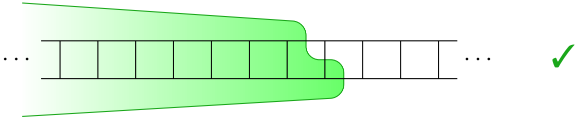
- 

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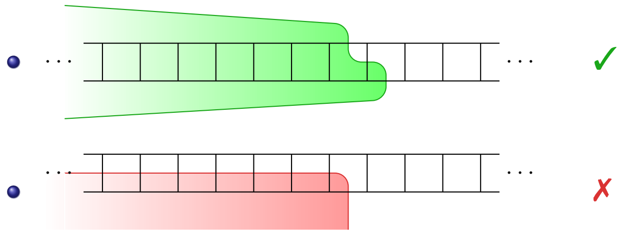

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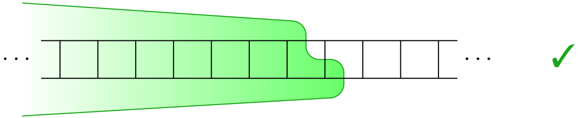
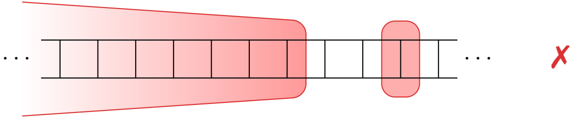
- 

- 

Tree sets

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A tree set is a set of cuts $\mathcal{C}$ such that

- $C \in \mathcal{C} \implies \overline{C} \in \mathcal{C}$,
- cuts in $\mathcal{C}$ are pairwise nested:

$$C \subseteq D \text{ or } C \subseteq \overline{D} \text{ or } D \subseteq C \text{ or } D \subseteq \overline{C} \text{ for all } C, D \in \mathcal{C},$$

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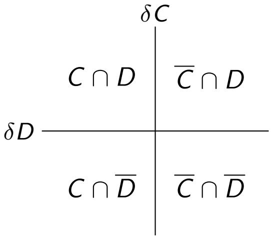

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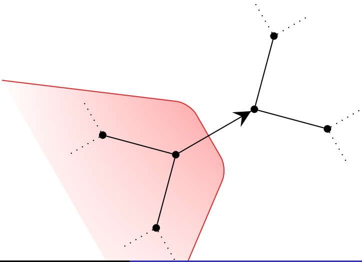

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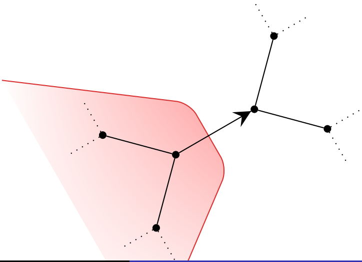
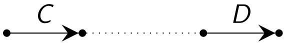
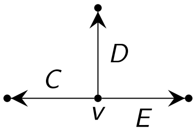




means $C \subseteq D$

vertices = equivalence classes of edges:



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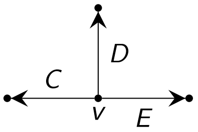

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For $C, D \in \mathcal{C}$ the relation $C \sim D$ is defined as follows:

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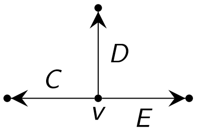

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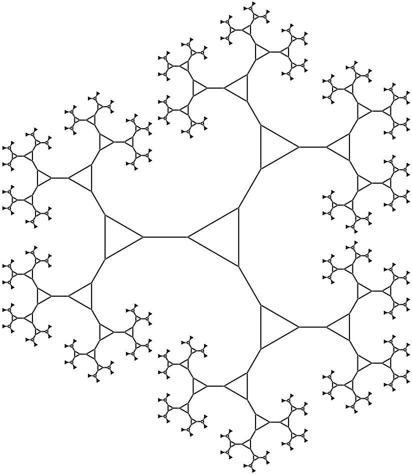
Proposition (Dunwoody, 1979)

The graph $T(\mathcal{C})$ is a tree, where

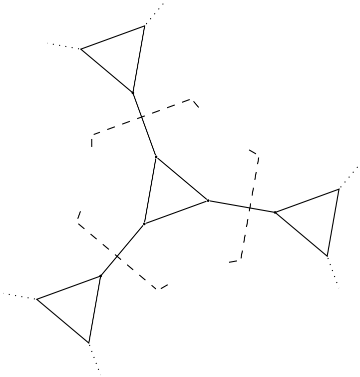
Vertices: $V(T(\mathcal{C})) = \{ [C] \mid C \in \mathcal{C} \}$,

Edges: $E(T(\mathcal{C})) = \{ \{[C], [\overline{C}]\} \mid C \in \mathcal{C} \}$.



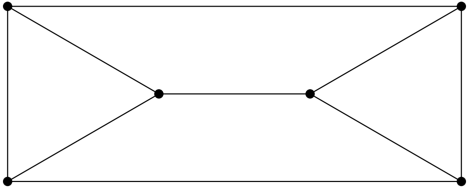
The Cayley graph of $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

Vertices in the structure tree



Three cuts in one equivalence class = one vertex in $T(\mathcal{C})$.

Example



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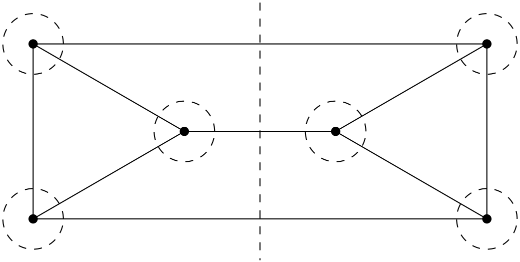

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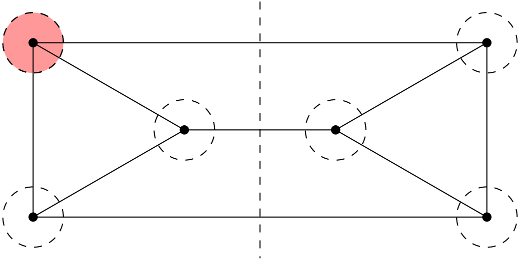

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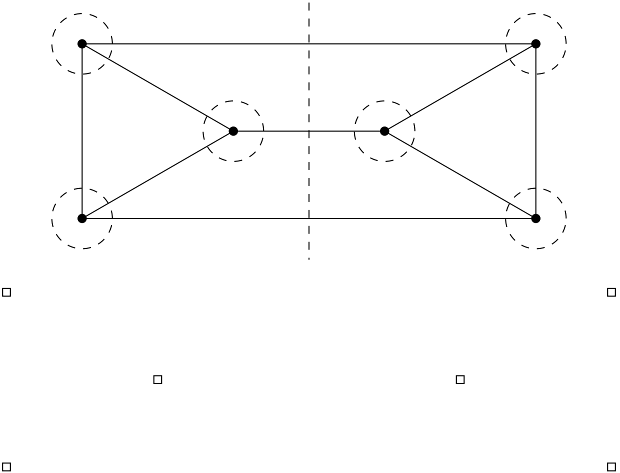

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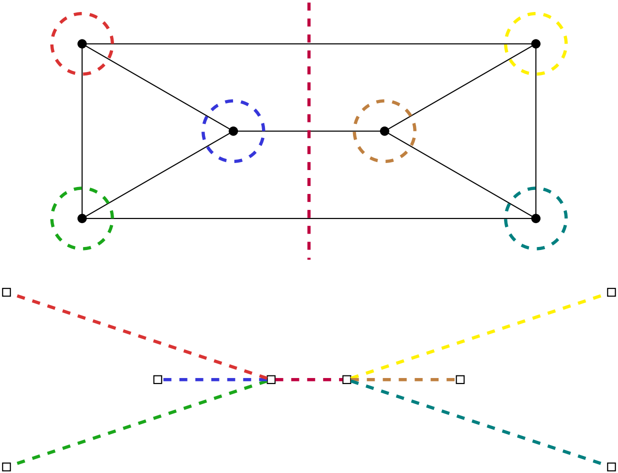

How to find a tree set?

Aims:

- find a structure tree for an arbitrary locally finite, connected graph Γ
- if Γ is tree-like, the structure tree should be locally finite
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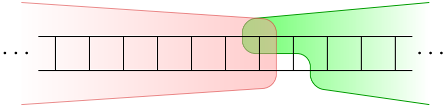

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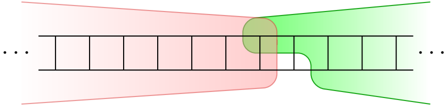

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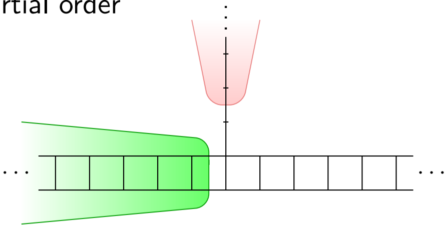

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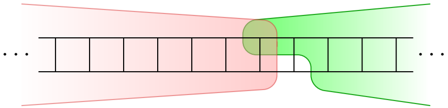

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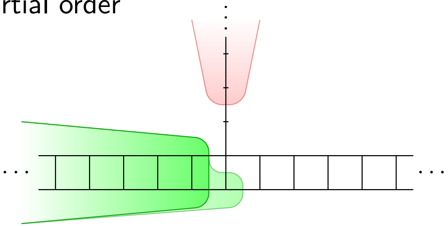

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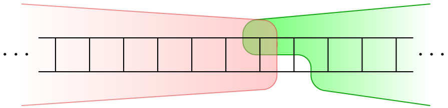
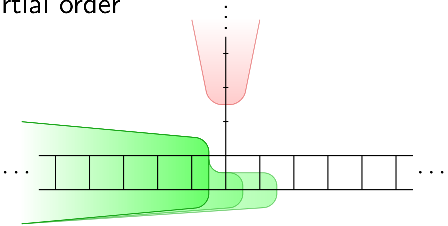


- not a discrete partial order



Aims:

- find a structure tree for an arbitrary locally finite, connected graph Γ
- if Γ is tree-like, the structure tree should be locally finite
  $\rightsquigarrow$ every bi-infinite geodesic should be split by some cut

First idea: take all cuts

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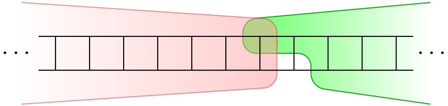

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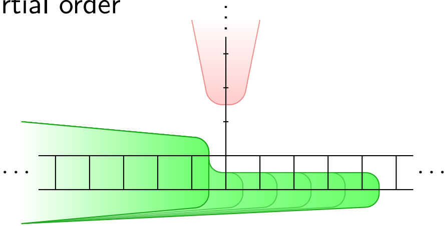
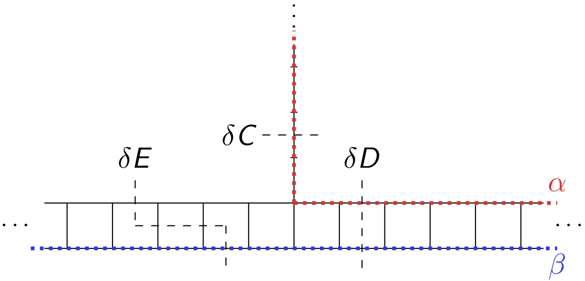


Minimal cuts = cuts which are minimal splitting a bi-infinite geodesic.



- $C \in \mathcal{C}_{\min}(\alpha)$
- $D \in \mathcal{C}(\alpha) \cap \mathcal{C}_{\min}$ but $D \notin \mathcal{C}_{\min}(\alpha)$
- $E \notin \mathcal{C}_{\min}$

Accessible graphs

A graph Γ is accessible iff $\exists\, K \in \mathbb{N}$ with $|\delta C| \leq K$ for all $C \in \mathcal{C}_{\mathsf{min}}$.

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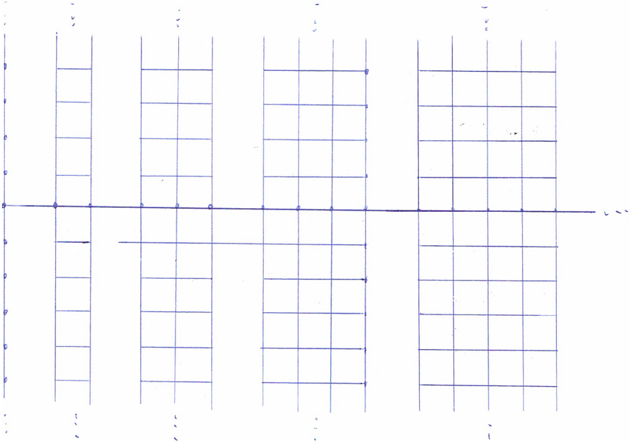

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Theorem (Thomassen, Woess, 1993)

G is accessible iff its Cayley graph Γ is accessible.

Theorem (Dunwoody, 1993)

There is a non-accessible group.

- $\rightsquigarrow$ There are non-accessible Cayley graphs.

But: every Cayley graph you can draw in a meaningful way is accessible.

- $\rightsquigarrow$ Tree-like Cayley graphs are accessible.

Accessible graphs

Lemma

The partial order $(\mathcal{C}_{\min}, \subseteq)$ is discrete iff Γ is accessible.

Accessible graphs

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The partial order $(\mathcal{C}_{\min}, \subseteq)$ *is discrete iff* Γ *is accessible.*

Lemma

Fix $K \in \mathbb{N}$ *and an edge* e *of* Γ. *There are only finitely many cuts* C *with* $e \in \delta C$ *and* $|\delta C| \leq K$.

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Fix $K \in \mathbb{N}$ and an edge e of Γ. There are only finitely many cuts C with $e \in \delta C$ and $|\delta C| \leq K$.

Proof (Thomassen, Woess, 1993).

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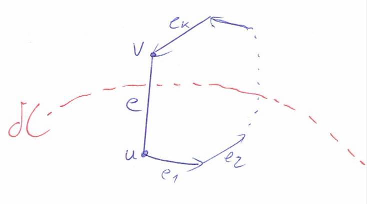

□

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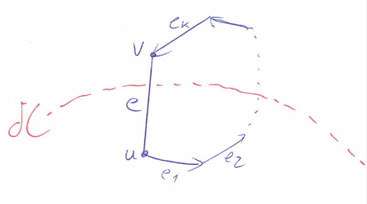
$$\mathcal{C}^K_{\Gamma,e} \subseteq \bigcup_{i=1}^{k} \mathcal{C}^{K-1}_{\Gamma - e, e_i}.$$



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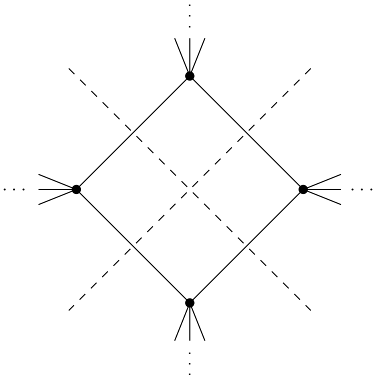
Minimal cuts still might not be nested:



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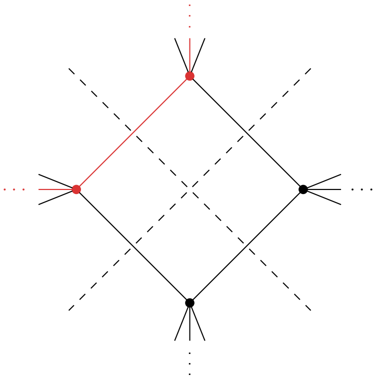

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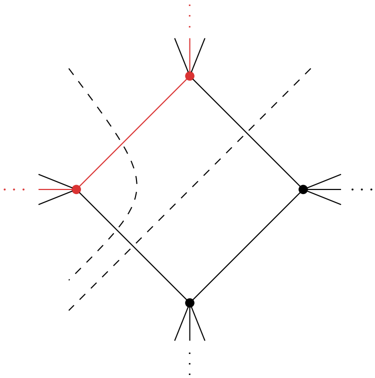
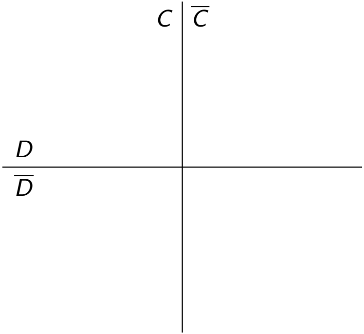


But, then we can switch to a subset.

Minimal Cuts

Let $C, D \in \mathcal{C}_{\min}$ not nested:



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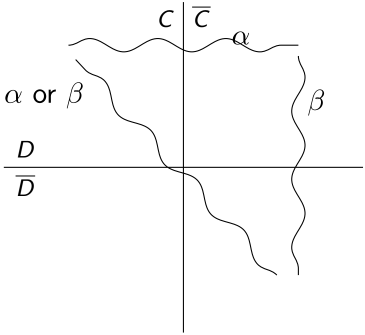

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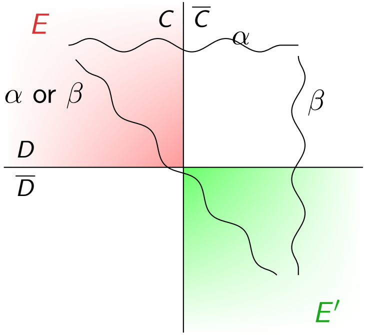

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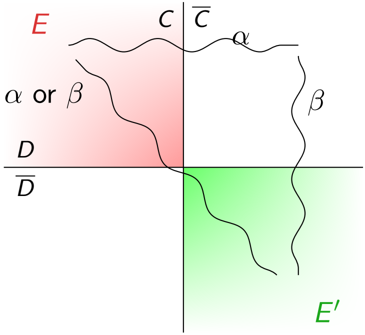

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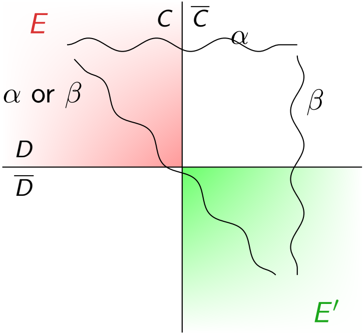

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$\rightsquigarrow$ take E and E' instead of C and D.

Optimal cuts

A cut C is optimal, if

- $C \in \mathcal{C}_{\min}(\alpha)$ for some bi-infinite geodesic α and
- the number of non-nested cuts is minimal among $\mathcal{C}_{\min}(\alpha)$.

Theorem (Diekert, W. 13)

For a tree-like Cayley graph Γ, the subset $\mathcal{C}_{\mathrm{opt}} \subseteq \mathcal{C}_{\mathsf{min}}$ satisfies:

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$G \backslash T(\mathcal{C}_{\mathrm{opt}})$ is the desired graph of groups (resp. a reduced subset of $\mathcal{C}_{\mathrm{opt}}$).

Back to the isomorphism problem: Roadmap for Proving the Main Lemma

Aim: find "small" isomorphism φ

- $\varphi(g) = g$ for $g \in G_P = \mathrm{Stab}(P)$
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Show:

- boundaries of minimal cuts are small
- equivalent cuts are not far apart

$\rightsquigarrow$ find representatives for $T(\mathcal{C}_{\mathrm{opt}})$ within $B(2^{2^{\mathcal{O}(N)}})$ (resp. $B(N^{\mathcal{O}(1)})$)

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Thank you!