TC^0 computations and the subgroup membership problem in nilpotent groups

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Outline

- Why circuit complexity for groups?
- Computing gcds
- Subgroup membership for nilpotent groups

Dehn's fundamental problems (and others)

Let G be a f. g. group, generated by a finite set $\Sigma = \Sigma^{-1} \subseteq G$.

- Word problem: Given $w \in \Sigma^*$. Question: Is $w = 1$ in G?
- Conjugacy problem: Given $v, w \in \Sigma^*$.
  Question: $\exists\, z \in G$ such that $zvz^{-1} = w$?
- (Uniform) Subgroup membership problem:
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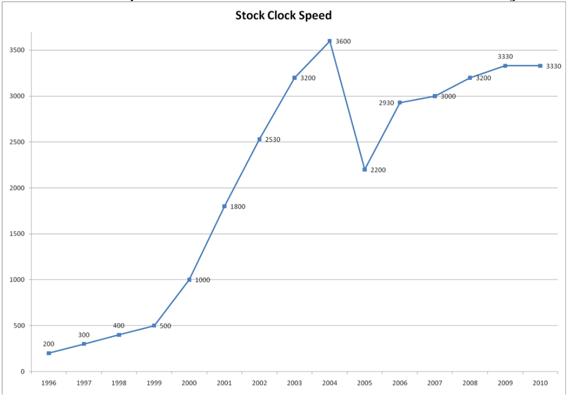

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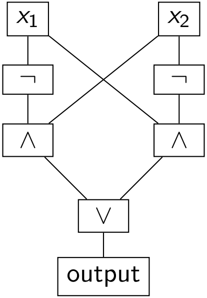

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size = number of gates
depth = longest path from input to output gate

NC = problems which can be solved by a family of circuits of polynomial size and polylogarithmic depth

= problems which can be solved by a parallel RAM with a polynomial number of processors in polylogarithmic time.

Inside NC:

- NC^i = solved by a family of circuits of depth $\mathcal{O}(\log^i n)$ and polynomial size with bounded fan-in (= in-degree) $\neg$, $\wedge$, $\vee$ gates.

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$$\mathsf{NC}^1 \subseteq \mathsf{LOGSPACE} \subseteq \mathsf{NC}^2 \subseteq \mathsf{NC}^3 \subseteq \cdots \subseteq \mathsf{NC} \subseteq \mathsf{P}.$$

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Theorem (Lipton, Zalcstein, 1977 / Simon, 1979)

The word problem of linear groups is in LOGSPACE.

"Proof": Given matrices $A_1, \ldots, A_n$, compute

$$\prod A_i \mod p$$

for sufficiently many primes p.

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Inside NC^1:

- AC^0 = solved by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates: $\mathrm{Maj}(w) = 1$ iff $|w|_1 \geq |w|_0$ for $w \in \{0,1\}^*$.

Theorem (Robinson, 1993)

The word problem of

- *Baumslag-Solitar groups* $\mathbf{BS}_{1,q}$ *and*
- *nilpotent groups*

are uniform TC^0*-complete.*

More problems in TC^0:

- conjugacy problem in $\mathbf{BS}_{1,q}$ (Diekert, Myasnikov, W., 2014)
- word problem in solvable linear groups (König, Lohrey, 2015)
- word and conjugacy problem in free solvable groups (Myasnikov, Vassileva, W., 2016)

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$$\begin{aligned} w \text{ represents } 0 \text{ in } \mathbb{Z} &\iff |w|_1 = |w|_0 \\ &\iff \mathrm{Maj}(w) \wedge \mathrm{Maj}(\neg w) \end{aligned}$$

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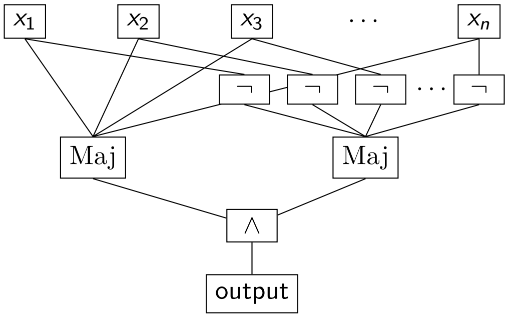

Arithmetic problems in TC^0

Iterated Addition

- input: n-bit numbers $r_1, \ldots, r_n$,
- compute $\sum_{i=1}^{n} r_i$.

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Integer Division

- input: n-bit numbers a, b,
- compute $\left\lfloor \frac{a}{b} \right\rfloor$.

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Theorem (Hesse, 2001)

Iterated Multiplication and Integer Division are in TC^0.

Reductions

- For a formal language $L \subseteq \{0,1\}^*$, $\mathsf{AC}^0(L)$ allows additionally oracle gates for L.
- $L' \in \mathsf{AC}^0(L)$ means L' is AC^0-reducible to L.
- Every problem in TC^0 is AC^0-reducible to Majority.

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The word problem of $\mathbb{Z}$ with generators $\{+1,-1\}$ is TC^0-complete.

Again, 1 encodes 1 and 0 encodes -1. For $u \in \{0,1\}^*$:

$$\begin{aligned}
\mathrm{Maj}(u) &\iff |u|_1 \geq |u|_0 \\
&\iff \bigvee_{0 \leq i \leq |u|} \left|u0^i\right|_1 = \left|u0^i\right|_0 \\
&\iff \bigvee_{0 \leq i \leq |u|} (u0^i \text{ represents } 0 \text{ in } \mathbb{Z})
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- $\mathsf{TC}^0 = \mathsf{AC}^0(\mathrm{WP}(\mathbb{Z})) \subseteq \mathsf{AC}^0(\mathrm{WP}(F_2))$
- $\mathsf{AC}^0(\mathrm{WP}(F_2)) \subseteq \mathsf{LOGSPACE}$

Overview: small circuit classes

AC^0	$= \mathsf{FO}(+,*)$	$\mathbb{Z}/n\mathbb{Z}$ with one monoid generator
ACC^0	$= \mathsf{FO}(+,*;\mathrm{Mod})$	finite solvable
TC^0	$= \mathsf{FO}(+,*;\mathrm{Maj})$	$\mathbb{Z}$, linear solvable (e. g. nilpotent), free solvable
$\mathsf{NC}^1 = \mathsf{AC}^0(\mathrm{WP}(A_5))$		finite non-solvable, regular languages
$\mathsf{AC}^0(\mathrm{WP}(F_2))$		virtually free, Baumslag-Solitar groups, RAAGs, free products
LOGSPACE		linear groups
NC		hyperbolic groups
P	polynomial time	compressed word problem of free groups, etc.

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Greatest Common Divisors

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Subgroup membership problem of $\mathbb{Z}$:

Given $a, a_1, \ldots, a_n \in \mathbb{Z}$, is $a \in \langle a_1, \ldots, a_n \rangle$?
With other words are there $x_1, \ldots, x_n \in \mathbb{Z}$ with

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Clearly, $a \in \langle a_1, \ldots, a_n \rangle$ iff $\gcd(a_1, \ldots, a_n) \mid a$.

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Observation

If $a_1, \ldots, a_n \in \mathbb{Z}$ are given in unary (a_i is represented by $\underbrace{11\cdots1}_{a_i \text{ many}}0\cdots0$), then the gcd can be computed in TC^0.

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Proof

Let $m = \max\{\,|a_i|\,\}$. For all $d \leq m$ do the following:

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Corollary

The subgroup membership problem of $\mathbb{Z}$ (where group elements are given as words over the generators) is in TC^0.

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Subgroup membership problem of $\mathbb{Z}^2$:

Given $a, b, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}$, is $(a, b) \in \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle$?
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$$(a_i', b_i') = (a_i, b_i) - \frac{a_i}{a_{n+1}}(a_{n+1}, b_{n+1})$$

(5) Set $b' = b - \frac{a}{a_{n+1}} b_{n+1}$ and check whether there are $x_1', \ldots, x_n' \in \mathbb{Z}$ such that $b' = x_1' b_1' + \cdots + x_n' b_n'$

Greatest Common Divisors as linear combinations

Question

Given $a_1, \dots, a_n \in \mathbb{Z}$ encoded in unary. Can $x_1, \dots, x_n \in \mathbb{Z}$ (in unary) with $d = x_1 a_1 + \cdots + x_n a_n$ be computed in TC^0?

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If $a_1, \ldots, a_n \in \mathbb{Z}$ are encoded in binary,

- it is not known whether the gcd can be computed in NC.
- finding the smallest $x_1, \ldots, x_n \in \mathbb{Z}$ is NP-complete (Majewski, Havas, 1994).

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Straightforward solution (try all possible values) does not work because there are too many:

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Straightforward solution (try all possible values) does not work because there are too many: Let $m = \max\{\,|a_i|\,\}$. There are $x_1, \dots, x_n \in \mathbb{Z}$ with $|x_i| \leq m/2$ – this is the best known upper bound (Majewski, Havas, 1994).

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$\rightsquigarrow$ m^n possible choices for the x_i to try.

However, if $n = 2$, there are only m^2 many values to try $\rightsquigarrow$ TC^0.

We can use this idea to compute $x_1, \ldots, x_n$ in TC^0:

Greatest Common Divisors as linear combinations

First, set $d_0 = 0$ compute

$$d_i = \gcd(a_1, \ldots, a_i) \quad \text{for } i = 1, \ldots, n$$

$$\rightsquigarrow \qquad d_i = \gcd(d_{i-1}, a_i).$$

For each i, compute integers y_i and z_i such that $d_i = y_i d_{i-1} + z_i a_i$.
Next compute

$$x_i = z_i \cdot \prod_{j=i+1}^{n} y_j$$

in TC^0 using iterated multiplication. Now, we have

$$x_1 a_1 + \cdots + x_n a_n = \gcd(a_1, \ldots, a_n).$$

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$\rightsquigarrow$ we have to make them smaller.

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If $n = 2$, this is easy:
Assume $a, b > 0$ and $ax + by = \gcd(a, b)$ with $x \geq b$. Set $p = \left\lfloor \frac{x}{b} \right\rfloor$ and replace

- x by $x - bp$ and
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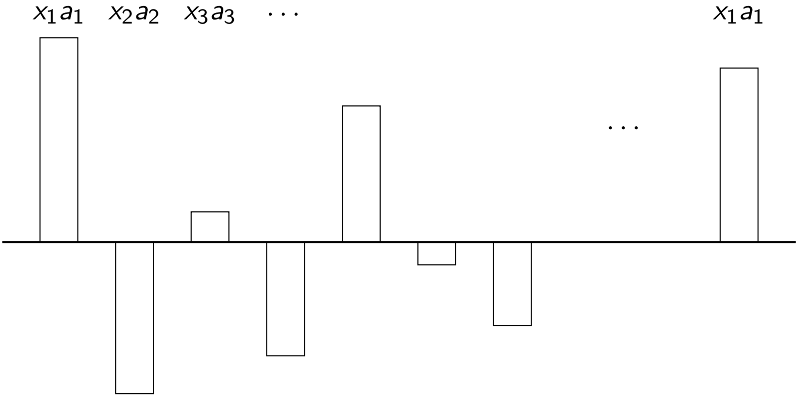
For which pairs?

Greatest Common Divisors as linear combinations



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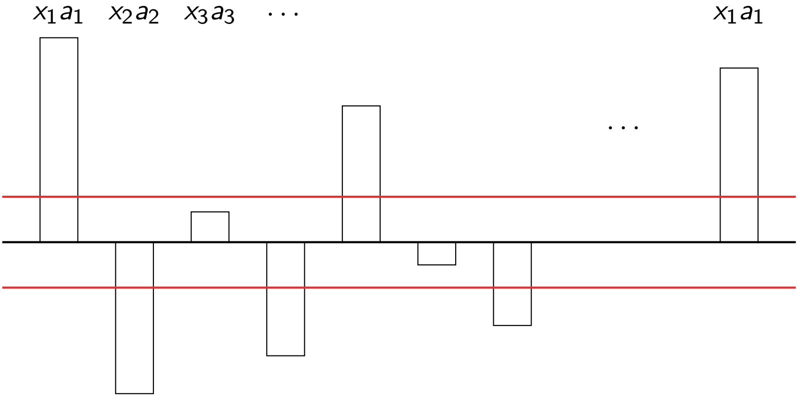

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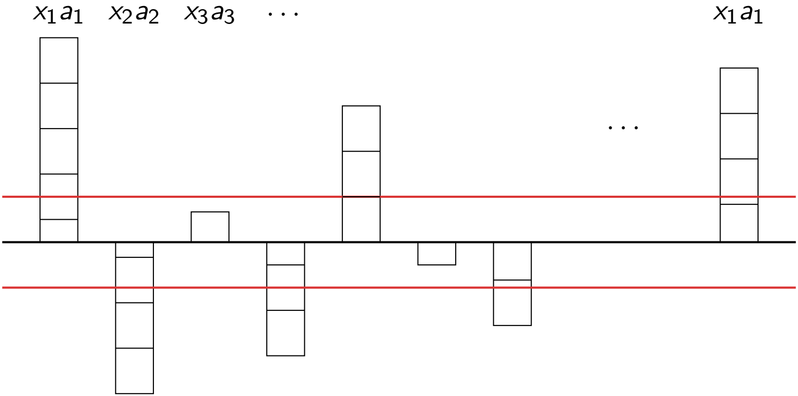

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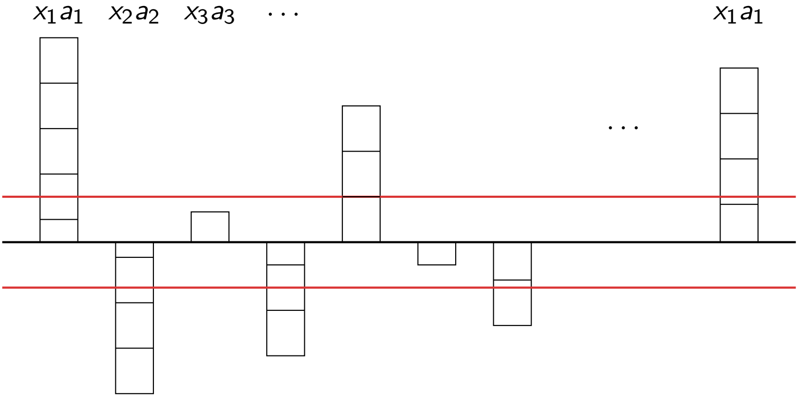

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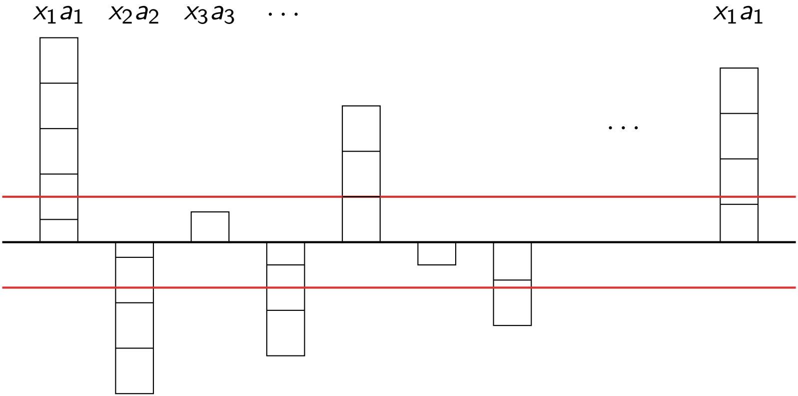

- Blocks of size $\max\{ a_i^2 \}$
- Using iterated addition, we can compute how many blocks from column i should go to column j in TC^0.
- Use idea for $n = 2$ to approximate blocks moved from column i to column j.

Greatest common divisors in TC^0

Theorem (Myasnikov, W., 2016)

There is a family of TC^0 circuits for the following problem: given $a_1, \ldots, a_n \in \mathbb{Z}$ encoded in unary, compute $x_1, \ldots, x_n \in \mathbb{Z}$ in unary with $d = x_1 a_1 + \cdots + x_n a_n$.

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Corollary

Let G *be a free abelian group. Then the subgroup membership problem for* G *is in* TC^0.

Definition

A group G is nilpotent of class c if

$$G = \Gamma_1(G) \geq \Gamma_2(G) \geq \cdots \Gamma_c(G) > \Gamma_{c+1}(G) = \{1\}$$

where $\Gamma_{i+1} = [\Gamma_i, G] = \left\langle x^{-1}g^{-1}xg \text{ for } x \in \Gamma_i, g \in G \right\rangle$.

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Theorem (Macdonald, Myasnikov, Nikolaev, Vassileva, 2015)

Let G be a nilpotent group. The (uniform) subgroup membership problem for G is in LOGSPACE.

The proof is based on so-called matrix reduction (Sims, 1994).

Mal'cev coordinates

Let G be a nilpotent group with Mal'cev basis $(a_1, \ldots, a_m) = \vec{a}$.

- Each $g \in G$ has a unique normal form

$$g = a_1^{x_1} \cdots a_m^{x_m} =: \vec{a}^{\vec{x}}$$

with $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{Z}^n$ (if there is torsion some of them are restricted $0 \leq x_i < e_i$) and such that

$$[a_i, a_j] \in \left\langle a_{\max\{\, i,j \,\}+1}, \ldots, a_m \right\rangle .$$

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- The product of two elements can be written in the same fashion

$$a_1^{x_1} \cdots a_m^{x_m} \cdot a_1^{y_1} \cdots a_m^{y_m} = a_1^{q_1} \cdots a_m^{q_m} .$$

  The exponents $q_1, \ldots, q_m$ are functions of $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ – if G is torsion-free they are polynomials.

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Fact

$$q_i(0, \ldots, 0, x_i, \ldots, x_m, y_1, \ldots, y_m) = x_i + y_i \pmod{e_i}$$

Matrix reduction

Let $(h_1, \ldots, h_n)$ be generators of a subgroup H. We associate a matrix of coordinates

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix},$$

where $(\alpha_{i1}, \ldots \alpha_{im})$ are the Mal'cev coordinate of h_i.

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We do "Gaussian elimination" until we reach a matrix satisfying (here, π_i is the position of the i-th pivot = first non-zero entry in row i):

(i) $\pi_1 < \pi_2 < \ldots < \pi_s$ (where s is the number of pivots),

(ii) $\alpha_{i\pi_i} > 0$, for all $i = 1, \ldots, n$,

(iii) $0 \leq \alpha_{k\pi_i} < \alpha_{i\pi_i}$, for all $1 \leq k < i \leq s$

(iv) if $e_{\pi_i} < \infty$, then $\alpha_{i\pi_i}$ divides e_{π_i}, for $i = 1, \ldots, s$.

(v) $H \cap \langle a_i, a_{i+1}, \ldots, a_m \rangle$ is generated by $\{h_j \mid \pi_j \geq i\}$, for all $1 \leq i \leq m$.

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Example: Matrix reduction

Let $G = \langle a_1, a_2, a_3 \mid [a_1, a_3] = [a_2, a_3] = 1,\ [a_1, a_2] = a_3 \rangle$ be the 3-dimensional Heisenberg group with Mal'cev basis (a_1, a_2, a_3).
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There are only a constant number of columns $\rightsquigarrow$ only a constant number of step and each can be done in TC^0.

Theorem (Myasnikov, W.)

Given $h_1, \ldots, h_n \in G$ (either as unary encoded Mal'cev coordinates or as words over the generators), Matrix reduction for the subgroup $\langle h_1, \ldots, h_n \rangle$ is in TC^0.

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Corollary (Myasnikov, W.)

Let G be a nilpotent group. The (uniform) subgroup membership problem for G is in TC^0.

More problems in TC^0

Uniform algorithms/circuits for r-generated class c nilpotent groups where r and c are fixed (Macdonald, Ovchinnikov, Myasnikov, W. – work in progress).

- Conjugacy problem
- Compute kernels and images of homomorphisms
- Compute centralizers
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Thank you!