Amenability of Schreier Graphs and Strongly Generic Algorithms for the Conjugacy Problem

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Overview

Part I

- Amenability of Schreier graphs
- Bounds for the number of elliptic elements of HNN extensions and amalgamated products

Part II

- The conjugacy problem for hyperbolic elements of HNN extensions and amalgamated products
- Strongly generic algorithms for the conjugacy problem

Amenability of Schreier Graphs

Special cases for fundamental groups of graphs of groups:

1. Amalgamated products

$$G = H \star_A K = \langle\, H, K \mid \varphi(a) = \psi(a) \text{ for } a \in A \,\rangle$$

   for groups H and K with a common subgroup A.

2. HNN extensions

$$G = \left\langle\, H, t_1, \ldots, t_k \;\middle|\; t_i a t_i^{-1} = \varphi_i(a) \text{ for } a \in A_i,\ i = 1, \ldots, k \,\right\rangle$$

   with stable letters $t_1, \ldots, t_k$ and an isomorphism $\varphi_i : A_i \to B_i$ for subgroups A_i and B_i of H.

H, K: vertex groups or base groups,
$A, A_1, \ldots, A_k$: edge groups or associated subgroups.

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$$\begin{aligned} \mathbf{BG}_{1,2} &= \left\langle\, a, b \;\middle|\; (bab^{-1})a(bab^{-1})^{-1} = a^2 \,\right\rangle \\ &= \left\langle\, \mathbf{BS}_{1,2}, b \;\middle|\; bab^{-1} = t \,\right\rangle \end{aligned}$$

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- Semidirect products
$H \rtimes F_k = \left\langle\, H, t_1, \ldots, t_k \;\middle|\; t_i h t_i^{-1} = \varphi_i(h), h \in H, i = 1, \ldots, k \,\right\rangle$

Schreier graphs

Schreier graph $\Gamma = \Gamma(G, P, \Sigma)$ of G with respect to a subgroup P and set of generators $\Sigma \subseteq G$:

- Vertices: $V(\Gamma) = P\backslash G = \{ Pg \mid g \in G \}$ = right cosets.

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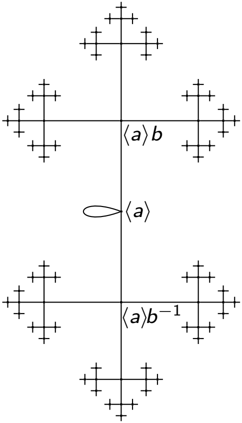
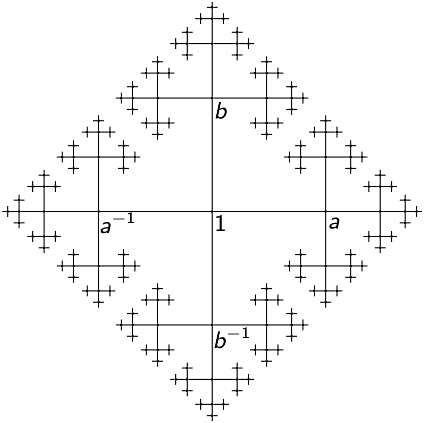
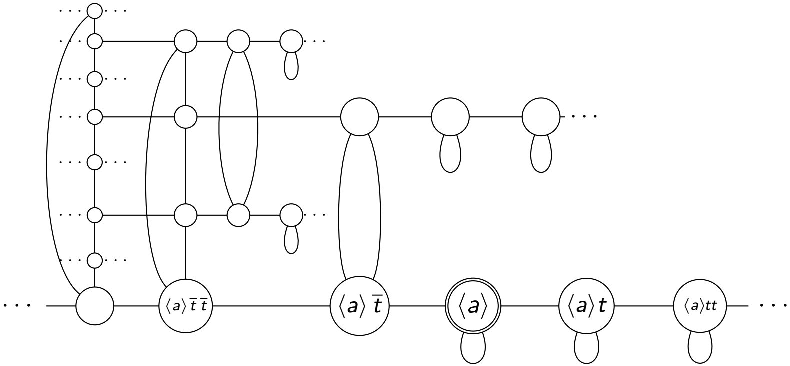
1-to-1 correspondence of words in Σ^* and paths starting at P.

- Every $2d$-regular graph is a Schreier graph (Gross 1977 for finite graphs, de la Harpe 2000 in general).
- Schreier graph $\Gamma(\langle a \rangle * \langle b \rangle, \langle a \rangle, \{ a, b, \bar{a}, \bar{b} \})$





- The Schreier graph $\Gamma(\mathbf{BS}_{1,2}, \langle a \rangle, \{ a, \bar{a}, t, \bar{t} \})$



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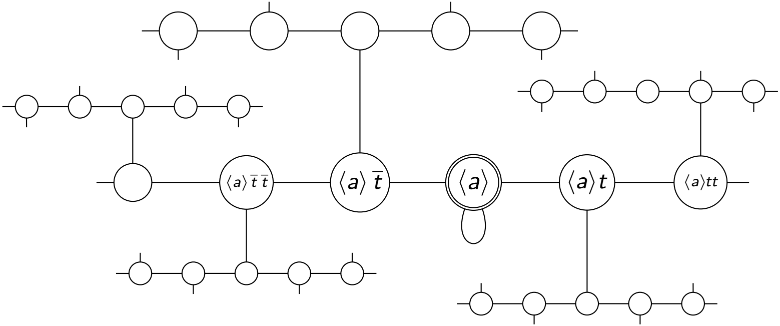

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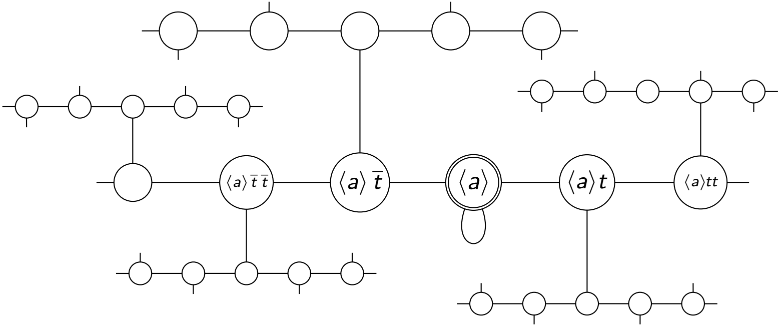

- Schreier graph $\Gamma(H \rtimes F_k, H, \Sigma) =$ Cayley graph $\Gamma(F_k, \{ 1 \}, \Sigma)$

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$$\left|\left\{ v \in V \;\middle|\; d(v, U) \leq k \right\}\right| \geq 2\,|U|\,.$$

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  If Γ is d-regular.

$$p^{(n)}(u, v) = \frac{\text{number of paths of length } n \text{ from } u \text{ to } v}{d^n}.$$

Amenability

Theorem (Kesten 1959, Gerl 1988, Gromov 1993,...)

Let $\Gamma = (V, E)$ be a d-regular undirected graph. The following statements are equivalent and define amenability:

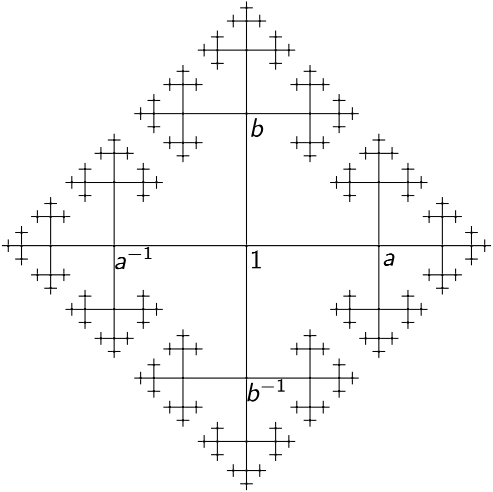
(1) *Γ satisfies the Gromov condition, i. e., there exists a map $f : V \to V$ such that $\sup_{v \in V} d(f(v), v) < \infty$ and $\left|f^{-1}(v)\right| \geq 2$ for all $v \in V$.*

(2) *Γ satisfies the doubling condition: there exists some $k \in \mathbb{N}$ such that for every finite $U \subseteq V$ we have*

$$\left|\left\{ v \in V \;\middle|\; d(v, U) \leq k \right\}\right| \geq 2\,|U|\,.$$

(3) *The random walk on Γ has exponentially decreasing return probability.*

The Cayley graph of the free group $F_{\{a,b\}}$ is non-amenable:



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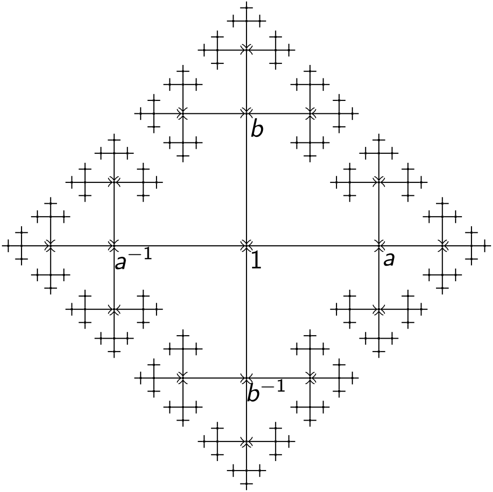

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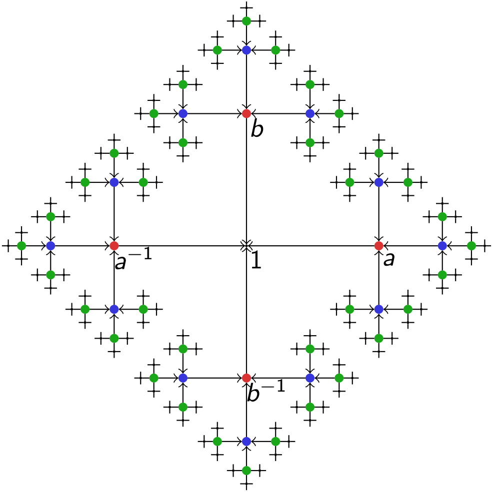

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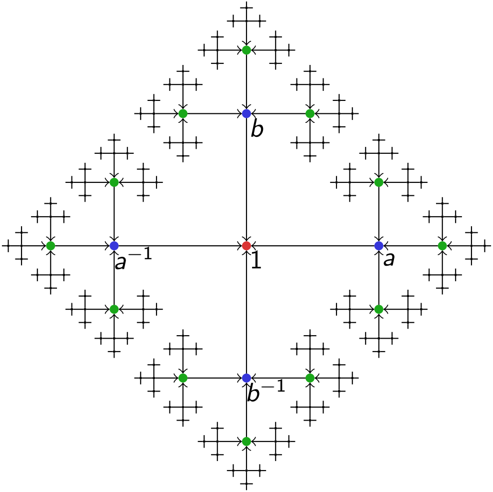
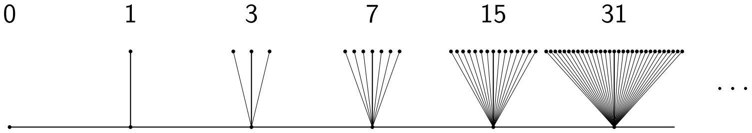
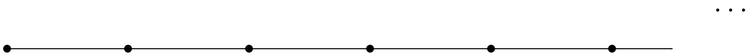


Amenability of locally finite graphs is not a quasi-isometry invariant!!!



The graph above satisfies the Gromov condition, but it is quasi-isometric to an amenable graph:



Examples

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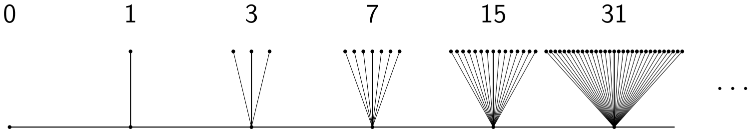

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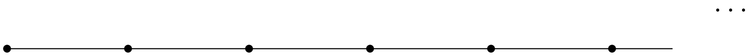

But: for d-regular graphs it is a quasi-isometry invariant.
$\rightsquigarrow$ invariant under change of generating set

Theorem (Diekert, Miasnikov, W. 2015)

Let $G = H \star_A K$ with $[H : A] \geq [K : A] \geq 2$ and $P \in \{H, K\}$ and let $\Sigma = \Sigma^{-1}$ generate G.
Then the Schreier graph $\Gamma(G, P, \Sigma)$ is non-amenable iff $[H : A] \geq 3$.

Characterization of Schreier Graphs

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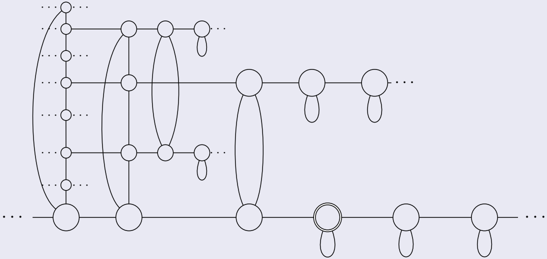
Theorem (Diekert, Miasnikov, W. 2015)

Let $G = \langle H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \rangle$ be an HNN extension and let $\Sigma = \Sigma^{-1}$ generate G.
The Schreier graph $\Gamma(G, H, \Sigma)$ is non-amenable iff both $[H : A] \geq 2$ and $[H : \varphi(A)] \geq 2$.

Example

Let $\mathbf{BS}_{p,q} = \left\langle\, a, t \;\middle|\; ta^p t^{-1} = a^q \,\right\rangle$ be the Baumslag-Solitar group with $1 \leq p \leq q$. Then the Schreier graph $\Gamma(\mathbf{BS}_{p,q}, \langle a \rangle, \{a, \bar{a}, t, \bar{t}\})$ is non-amenable iff $p \neq 1$.



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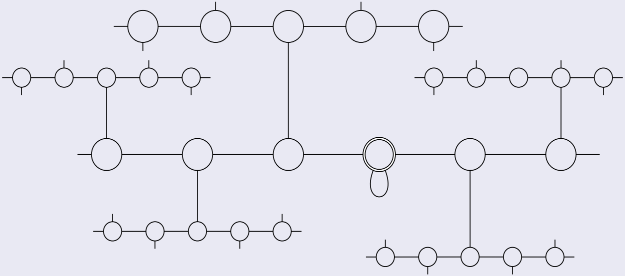

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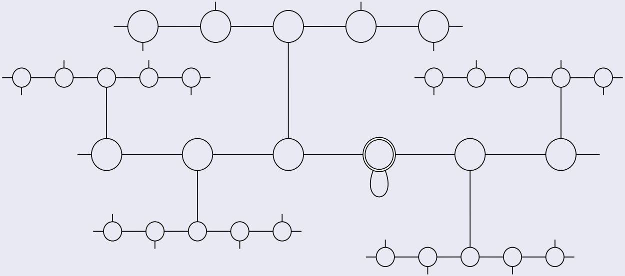

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The Schreier graph $\Gamma(\mathbf{BG}_{1,2}, \mathbf{BS}_{1,2}, \{a, \bar{a}, b, \bar{b}\})$ is non-amenable.

Recall: $\mathbf{BG}_{1,2} = \left\langle\, \mathbf{BS}_{1,2}, b \;\middle|\; bab^{-1} = t \,\right\rangle$.

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- If $k = 1$:

$$H \rtimes F_k = \langle\, H, t \;\big|\; tht^{-1} = \varphi(h) \text{ for } h \in H \,\rangle$$

and $[H : H] = [H : \varphi(H)] = 1$.

- If $k \geq 2$:

$$H \rtimes F_k = \langle\, G, t_k \;\big|\; t_k a t_k{}^{-1} = \varphi_k(a) \text{ for } a \in A_k \,\rangle$$

for $G = \langle\, H, t_1, \ldots, t_{k-1} \;\big|\; t_i a t_i{}^{-1} = \varphi_i(a) \text{ for } a \in A_i \,\rangle$
and $[G : A_k] = [G : \varphi(A_k)] = \infty$.

Proof for amalgamated products

Theorem (Diekert, Miasnikov, W. 2015)

Let $G = H \star_A K$ with $[H:A] \geq [K:A] \geq 2$ and $P \in \{H, K\}$ and let $\Sigma = \Sigma^{-1}$ generate G.
Then the Schreier graph $\Gamma(G, P, \Sigma)$ is non-amenable iff $[H:A] \geq 3$.

Proof

For the only-if direction we assume $[H:A] = [K:A] = 2$.
$\rightsquigarrow$ A is normal in G and $G/A = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = D_\infty$.

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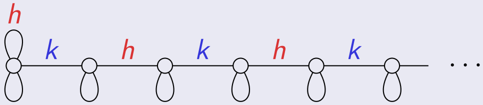
Assume $\Sigma \subseteq A \cup \{h, k\}$ for some $h \in H$, $k \in K$. Then the Schreier graph $\Gamma(G, H, \Sigma)$ is amenable:



Proof for amalgamated products

Lemma (Normal forms for amalgamated products)

Fix transversals $C \subseteq H$ and $D \subseteq K$ for cosets of A in H and K with $1 \in C \cap D$ s. t. the decompositions

$$H = AC, \qquad K = AD$$

are unique.

Every group element $g \in G = H \star_A K$ can be uniquely written as

$$g =_G x_0 \cdots x_k$$

for some $k \in \mathbb{N}$, $x_0 \in H \cup K$ such that for all $1 \leq i \leq k$ we have

$$x_i \in C \cup D \setminus \{1\};$$
$$x_{i-1} \in H \iff x_i \in K.$$

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Let $[H : A] \geq 3$. We show the Gromov condition (1).
Let $f : P\backslash G \to P\backslash G$ as follows:

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Fix $c \neq c' \in C \setminus \{1\}$ and $d \in D \setminus \{1\}$.

- For a normal form $x_0 \cdots x_k$ with $x_k = d$ and $x_{k-1} \in \{c, c'\}$, set $f(Px_0 \cdots x_k) = Px_0 \cdots x_{k-2}$.
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✓ $\sup \{ d(f(Pw), Pw) \mid Pw \in P\backslash G \} < \infty$.

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✓ $\sup \{ d(f(Pw), Pw) \mid Pw \in P\backslash G \} < \infty$.

✓ For every normal form w, either wcd and $wc'd$ or wdc and wdc' are normal forms. Hence, $\left|f^{-1}(Pw)\right| \geq 2$ for all $w \in G$.

Action on the Bass-Serre tree

A fgogog G acts on its Bass-Serre tree.

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Consequence

- $\{$ elliptic elements $\} = \bigcup_{g \in G} g(H \cup K)g^{-1}$, or
- $\{$ elliptic elements $\} = \bigcup_{g \in G} gHg^{-1}$.
- $\{$ Hyperbolic elements $\} = G \setminus \{$ elliptic elements $\}$.

Hyperbolic elements form a strongly generic subset if . . .

$S \subseteq \Sigma^*$ is called generic if $\frac{|\Sigma^n \setminus S|}{|\Sigma^n|} \to 0 \quad \text{for } n \to \infty$.

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Theorem (Diekert, Miasnikov, W. 2015)

- *Let $G = H \star_A K$ be an amalgamated product such that $[H : A] \geq 3$ and $[K : A] \geq 2$, or let*
- $G = \langle\, H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \,\rangle$ *be an HNN extension with $[H : A] \geq 2$ and $[H : \varphi(A)] \geq 2$.*

Then the set of words representing hyperbolic elements in G is strongly generic in Σ^*.

Proof

Under the hypotheses of the characterization theorems:

- $\{ w \in \Sigma^* \mid w \in H \cup K \}$ is strongly generic.
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Assume $\Sigma \subseteq H \cup K$ (resp. $\Sigma \subseteq H \cup \{t, \bar{t}\}$).

Then $w \in \Sigma^*$ represents an elliptic group element iff there is some cyclic permutation $w' = w_2 w_1$ of $w = w_1 w_2$ with $w' \in H \cup K$.

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There are only $|w|$ cyclic permutations:

$$\begin{aligned} |\{ w \in \Sigma^n \mid w \text{ elliptic} \}| &\leq n \cdot |\{ w \in \Sigma^n \mid w \in H \cup K \}| \\ &\leq n \cdot 2^{\varepsilon n} \leq 2^{\varepsilon' n} \qquad \text{for } n \text{ large enough.} \end{aligned}$$

$\rightsquigarrow$ hyperbolic elements form a strongly generic set.

The conjugacy problem in HNN extensions and amalgamated products

Dehn's fundamental problems

Let G be generated by a finite set Σ with $\Sigma = \Sigma^{-1}$, i. e., there is an epimorphism

$$\eta : \Sigma^* \to G.$$

Write $\overline{a}$ for $a^{-1} \in \Sigma$.

- Word problem: Given $w \in \Sigma^*$. Question: Is $w = 1$ in G?
- Conjugacy problem: Given $v, w \in \Sigma^*$. Question: $v \sim w$?

  $(\exists\, z \in G$ such that $zvz^{-1} = w$?)

Examples

- Baumslag-Solitar groups $\mathbf{BS}_{p,q}$: Conjugacy problem is decidable in Logspace (W. 2015).

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- Semidirect products
  $H \rtimes F_k = \left\langle\, H, t_1, \ldots, t_k \;\middle|\; t_i h t_i^{-1} = \varphi_i(h), h \in H, i = 1, \ldots, k \,\right\rangle$

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Theorem (Miller 1968)

There is a group $F_n \rtimes F_k$ with undecidable conjugacy problem.

Theorem (Bogopolski, Martino, Ventura 2010)

There is a group $\mathbb{Z}^4 \rtimes F_k$ with undecidable conjugacy problem.

Strongly generic algorithms

$S \subseteq \Sigma^*$ is called generic if $\dfrac{|\Sigma^n \setminus S|}{|\Sigma^n|} \to 0 \quad$ for $n \to \infty$.

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A problem $\mathcal{P}$ is (strongly) generically decidable (in polynomial time) if there is a partial algorithm $\mathcal{A}$ and a strongly generic set S such that

1. $\mathcal{A}$ solves $\mathcal{P}$ (in polynomial time) on all inputs from S.
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3. If $\mathcal{A}$ gives an answer, then the answer must be correct.

The algorithm $\mathcal{A}$ never fools and gives an answer (in polynomial time) on "almost all" random inputs.

Strongly generic algorithms

"Trivial" generic algorithm for HNN extensions (Kapovich, Miasnikov, Schupp, Spilrain 2003):

$$G = \langle\, H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \,\rangle$$

Compute the image under $\varphi : G \to \langle t \rangle = G/\langle\langle H \rangle\rangle$ (count the number of letters t).

- if $\varphi(v) \neq \varphi(w)$, then v and w are not conjugate,
- otherwise, nothing is known.

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Theorem (Borovik, Miasnikov, Remeslennikov 2005)

The conjugacy problem of Miller's group $F_n \rtimes F_k$ is strongly generically decidable in polynomial time.

Solving the conjugacy problem of hyperbolic elements

Lemma (Collins' Lemma)

Let $G = \langle H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \rangle$ *and let* $v, w \in \Sigma^*$ *be*

- *cyclically Britton-reduced, (no factor* tat^{-1} *or* $t^{-1}bt$ *in* vv *and* ww *for any* $a \in A$ *or* $b \in \varphi(A)$*),*
- *representing hyperbolic group elements.*

Then

$$v \sim w \iff \text{there is a cyclic permutation } w_2 w_1 \text{ of } w = w_1 w_2$$
$$\text{and } a \in A \text{ such that } v = a w_2 w_1 a^{-1}.$$

Groups with more than one end

Observation

Let

$$G = \langle\, H, t_1, \ldots, t_k \mid t_i a t_i^{-1} = \varphi_i(a) \text{ for } a \in A_i,\ i = 1, \ldots, k \,\rangle$$

with A_i finite for all i. If the the word problem of G is decidable, then the conjugacy problem of G is decidable for hyperbolic elements.

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Proof.

Input: v, w

- Apply Britton reductions cyclically.
- Simply test for all $a \in \bigcup_i A_i$ and all cyclic permutations $w_2 w_1$ of w whether $v = a w_2 w_1 a^{-1}$.

□

Corollary

Let G be a finitely generated group with more than one end. If the word problem of G is decidable in polynomial time, then the conjugacy problem of G is decidable in polynomial time in a strongly generic setting.

Proof.

By Stallings' Structure Theorem, G splits over a finite subgroup. There are two cases:

- G is virtually cyclic $\rightsquigarrow$ conjugacy problem in linear time.
- Otherwise, hyperbolic elements form a strongly generic set.

□

HNN extenstions of free abelian groups

Theorem

Let

$$G = \langle\, H, t_1, \ldots, t_k \mid t_i a t_i{}^{-1} = \varphi_i(a) \text{ for } a \in A_i,\ i = 1, \ldots, k \,\rangle$$

with H finitely generated free abelian. Then for hyperbolic elements, the conjugacy problem of G is decidable in polynomial time.

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with H finitely generated free abelian. Then for hyperbolic elements, the conjugacy problem of G is decidable in polynomial time.

The proof is based on:

Theorem (Frumkin 1977, von zur Gathen, Sieveking 1978)

Given a system of linear equation with integer coefficients, it can be determined in polynomial time whether it has an integral solution and, if so, the solution can be computed in polynomial time.

HNN extenstions of free abelian groups

Proof

Choose bases for H and for the A_i. This defines integer matrices $M_i^{(1)}, M_i^{(-1)}$ for the inclusions

$$\mathrm{id} : A_i \to H, \qquad \varphi_i : A_i \to H.$$

- Subgroup membership problem for A_i (resp. $\varphi(A_i)$) reduces to a system of linear integer equations.
- Britton reductions $t_i g t_i^{-1} \to \varphi_i(g)$ in polynomial time.
- Compute cyclically Britton-reduced words in polynomial time.

HNN extenstions of free abelian groups

Proof (Cont.)

Apply Collins' Lemma:

- Check all cyclic permutations.
- Let $v = t_{i_1}^{\varepsilon_1} g_1 \cdots t_{i_n}^{\varepsilon_n} g_n, \quad w = t_{i_1}^{\varepsilon_1} h_1 \cdots t_{i_n}^{\varepsilon_n} h_n$ be cyclically reduced with $g_i, h_i \in H$. Then there is some $a \in \bigcup_i A_i$ with $ava^{-1} =_G w$ iff the system of equations

$$M_{i_j}^{(\varepsilon_j)} x_j - M_{i_{j+1}}^{(\varepsilon_j)} x_{j+1} + g_j = h_j \qquad \text{for } 1 \le j \le n,$$

has an integral solution $x_1, \ldots, x_n$.

HNN extenstions of free abelian groups

Corollary

$$G = \langle\, H, t_1, \ldots, t_k \mid t_i a {t_i}^{-1} = \varphi_i(a) \text{ for } a \in A_i,\ i = 1, \ldots, k \,\rangle$$

with H finitely generated free abelian. The conjugacy problem of G is decidable in polynomial time on a strongly generic set.

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- $G = \langle\, H, t \,|\, tat^{-1} = \varphi(a) \text{ for } a \in H \,\rangle$:
  for $g, h \in H$ (i. e., g, h elliptic):

  $$g \sim h \text{ iff } \exists i \in \mathbb{N} \text{ with } g = \varphi^i(h) \text{ or } h = \varphi^i(g)$$

  $\rightsquigarrow$ orbit problem for rational matrices. Dedicable in polynomial time (Kannan, Lipton 1986).
  If $\varphi(H) = H$, see also Cavallo, Kahrobaei 2014.

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with H finitely generated free abelian. The conjugacy problem of G is decidable in polynomial time on a strongly generic set.

Application

The conjugacy problem of the $\mathbb{Z}^4 \rtimes F_n$ group with undecidable conjugacy problem (Bogopolski, Martino, Ventura 2010) is strongly generically in polynomial time.

Theorem (Diekert, Miasnikov, W. 2014)

The conjugacy problem of $\mathbf{BG}_{1,2}$ *is decidable in polynomial time for hyperbolic elements.*

Baumslag Group

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Conjugacy in the Baumslag group $\mathbf{BG}_{1,2}$ can be solved in polynomial time in a strongly generic setting by some algorithm which always stops and which has non-elementary average time complexity.

Conjecture

The conjugacy problem of $\mathbf{BG}_{1,2}$ is non-elementary on average.

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Hence, there are natural problems / algorithms where average case complexity is meaningless! Because average case is not better than worst case and the worst case is useless.

Difficulty of the word problem in $\mathbf{BG}_{1,2}$

τ = tower function: $\tau(0) = 0$, $\tau(n+1) = 2^{\tau(n)}$.

Solving the word problem using Britton reductions:

$$ba^k b^{-1} \to t^k \qquad\qquad b^{-1} t^k b \to a^k$$

leads to non-elementary blow-up.

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$$\begin{aligned} w_{n+1} &:= b \cdot w_n \cdot a \cdot w_n^{-1} \cdot b^{-1} \\ &= b \cdot t^{\tau(n)} \cdot a \cdot t^{-\tau(n)} \cdot b^{-1} \\ &= b \cdot a^{\tau(n+1)} \cdot b^{-1} \\ &= t^{\tau(n+1)} \end{aligned}$$

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$|w_n| \in 2^{\Theta(n)}$, but w_n is a huge compression for the number $\tau(n)$.

HNN extenstions of free abelian groups

For the word problem: use power circuits for high compression.

Algorithm for conjugacy for hyperbolic elements

- Reduce words cyclically using the algorithm by Miasnikov, Ushakov, Won.
- Check all cyclic permutations.
- For each cyclic permutation, compute a "cyclic" normal form.
- Use the word problem to check normal forms for equality.

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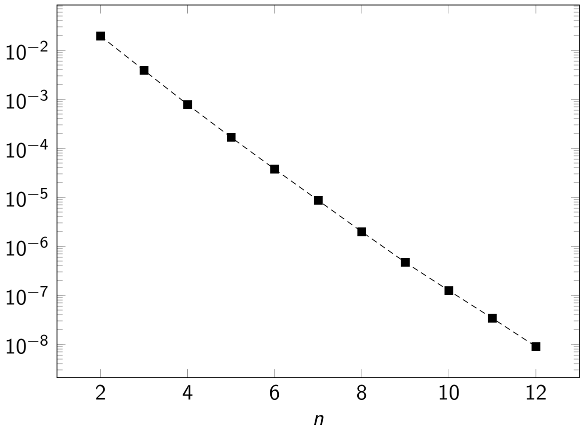
Problem for elliptic elements:

$$a^r t^m \sim a^s t^q \iff m = q \text{ and } \exists k \in \mathbb{N} : 0 \leq k < m \text{ such that}$$
$$r \cdot 2^k \equiv s \mod 2^m - 1$$

r, m, s, q extremely huge numbers given by power circuits.



Portion of reduced words $w \in H$ over the alphabet $\{a, b, \overline{a}, \overline{b}\}$ with $|w|_b + |w|_{\overline{b}} = 2n$, sampling $11 \cdot 10^9$ words.

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Thank you!