The power word problem

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- Power word problem (PowerWP):
  Given $p_1, \dots, p_k \in \Sigma^*$ and $x_1, \dots, x_k \in \mathbb{Z}$.
  Question: $p_1^{x_1} \cdots p_k^{x_k} = 1$ in G?

$$\textit{Is} \qquad b^{123} (b\, a\, a)^{123} a^{-246} b^{-123} (b\, a)^{-123} a^{123} = 1 \qquad ?$$

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- tool for the knapsack problem in RAAGs (Lohrey, Zetsche, 2015) (Given $p_1, \ldots, p_k, w \in \Sigma^*$, $\exists\, x_1, \ldots, x_k \in \mathbb{N}$ with $p_1^{x_1} \cdots p_k^{x_k} = w$?)

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- better understanding of the compressed word problem:
  - lower bounds
  - better upper bounds in the special case

Word problems of free groups

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The power word problem for free groups is in $\mathrm{AC}^0(\mathrm{WP}(F_2))$.

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The proof consists of three steps:

- Preprocessing
- Make exponents small
- Solve regular word problem

Let $F = F(\{a, b\})$ be the free group. Write $\overline{a}$ for a^{-1}.

Example 1

$$(a\, b)^{1000}\, a\, b^{-100}\, b^{100}\, a\, b^{-100}\, b^{100}\, \overline{a}\, \overline{a}\, (a\, b)^{-1000}$$

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Example 4

$$(b\,a\,a\,\overline{a}\,b\,a)^{500}\,(b)^{2}\,(\overline{b}\,\overline{b}\,\overline{a}b)^{999}\,(\overline{b}\,\overline{a}\,\overline{b}\,\overline{b}\,a\,b)^{1}(a\,b)^{-1}$$

$\Omega \subseteq \Sigma^+$ is set of non-empty words p with

(1) p is cyclically reduced,
(2) p is primitive,
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$$\Omega = \{ a, b, ab, a\overline{b}, aab, aa\overline{b}, \dots \}$$

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Lemma

Let $p, q \in \Omega$ and v a factor of p^x and w a factor of q^y.
If $vw = 1$ in F and $|v| = |w| \geq |p| + |q| - 1$, then $p = q$.

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- By (2), $|p| = |q|$.
- By (3), since p is a factor of w^{-1}, we get $p = q$. $\square$

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Now we have a "nice" instance

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But: cannot delete them entirely:

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$$a^{100} b\, a^{-100} \overline{b} \neq 1, \text{ but } a^{0} b\, a^{0} \overline{b} = 1$$

Nor down to 1:

$$a^{100} (\overline{a}\, b\, a)^1 a^{-100} \overline{b} \neq 1 \text{ but } a^{1} (\overline{a}\, b\, a)^1 a^{-1} \overline{b} = 1$$

Make exponents small

For $p \in \Omega$ rite $w = u_0 p^{y_1} u_1 \cdots p^{y_m} u_m$ such that no u_i contains p^x.

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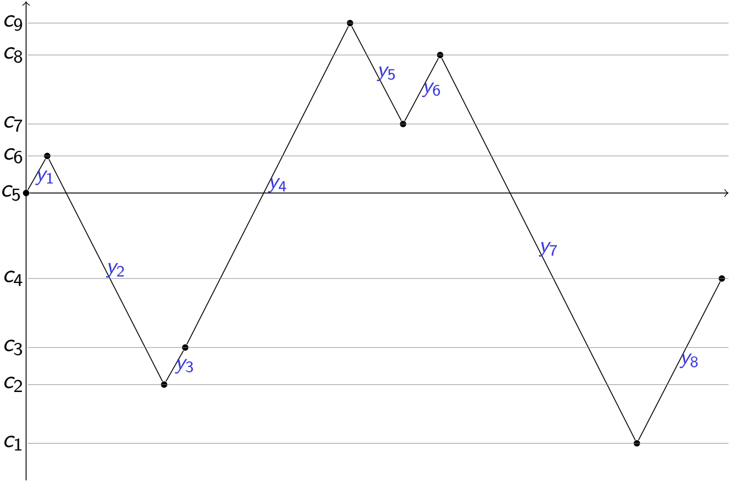

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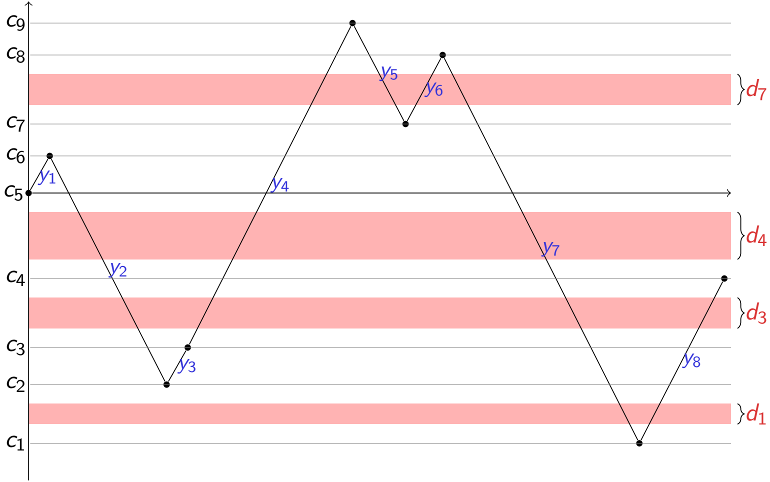

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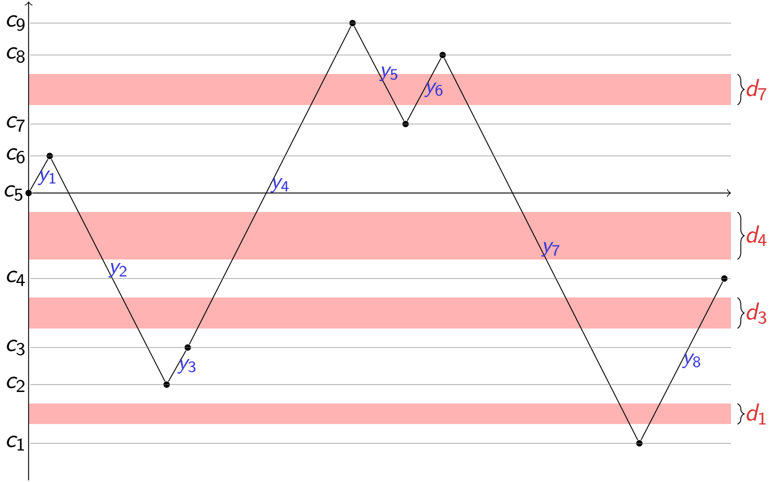

Define $\mathcal{S}(w) = u_0 p^{z_1} u_1 \cdots p^{z_m} u_m$ where $z_i = y_i - \operatorname{sign}(y_i) \cdot \sum_{j \in C_i} d_j$

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Proposition

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Proof of the main theorem.

- Preprocessing gives a "nice word" $w = s_0 p_1^{x_1} s_1 \cdots p_n^{x_n} s_n$.
- For all $p \in \Omega$ which appear in w, compute $\mathcal{S}(w)$ in parallel (iterated addition $\rightsquigarrow$ in TC^0).
- Yields a word of polynomial length $\rightsquigarrow$ ordinary word problem.

Further results on the power word problem

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Let G be f.g. and $H \leq G$ of finite index. Then

$$\textsc{PowerWP}(G) \leq_{\mathrm{m}}^{\mathsf{NC}^1} \textsc{PowerWP}(H).$$

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For every f.g. abelian group G, $\textsc{PowerWP}(G \wr \mathbb{Z})$ is in TC^0.

The power word problem in wreath products

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Let G be either

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- *f.g. free of rank ≥ 2.*

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For comparison:

- $\mathrm{WP}(G \wr \mathbb{Z})$ is in LOGSPACE (resp. NC^1)
- $\textsc{CompressedWP}(G \wr \mathbb{Z})$ is PSPACE-complete (Lohrey 2019, unpublished)

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Proof idea.

Show $\text{CNF-}\textsc{UnSAT} \leq \textsc{PowerWP}(G \wr \mathbb{Z})$:

- Every formula can be "simulated" in G (Barrington 89)
- Test all valuations "in parallel" in $G^{(\mathbb{Z})} \leq F_2 \wr \mathbb{Z}$

Open Questions

- What if we allow nested exponents:

$$\left(b^{13}\overline{a}\;\left((b\,a^{8}a)^{13}a^{-26}b^{-13}\right)^{12}\right)^{16}\left((\overline{b}\,\overline{a})^{13}a^{13}\right)^{20}$$

  - Conjecture: for constant nesting depth in $\mathsf{AC}^0(\mathrm{WP}(F_2))$.
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  - $(G \wr \mathbb{Z})$ for G non-abelian, but not free nor finite, non-solvable (e. g. G nilpotent)?
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Thank you!