Amenability of Schreier Graphs and Strongly Generic Algorithms for the Conjugacy Problem

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Overview

- The conjugacy problem in HNN extensions and amalgamated products
- Strongly generic algorithms
- Amenability of Schreier graphs
- Applications to the conjugacy problem

The conjugacy problem in HNN extensions and amalgamated products.

Dehn's fundamental problems

Let G be a group, generated by a finite set Σ with $\Sigma = \Sigma^{-1} \subseteq G$.

- Word problem: Given $w \in \Sigma^*$. Question: Is $w = 1$ in G?
- Conjugacy problem: Given $v, w \in \Sigma^*$. Question: $v \sim w$?

  ($\exists\, z \in G$ such that $zvz^{-1} = w$?)

Graph of groups

Special cases for fundamental groups of graphs of groups:

1. Amalgamated products

$$G = H \star_A K = \langle\, H, K \mid \varphi(a) = \psi(a) \text{ for } a \in A \,\rangle$$

   for groups H and K with a common subgroup A.

2. HNN extensions

$$G = \left\langle\, H, t_1, \ldots, t_k \;\middle|\; t_i a t_i^{-1} = \varphi_i(a) \text{ for } a \in A_i,\ i = 1, \ldots, k \,\right\rangle$$

   with stable letters $t_1, \ldots, t_k$ and an isomorphism $\varphi_i : A_i \to B_i$ for subgroups A_i and B_i of H.

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H, K: vertex groups or base groups
$A, A_1, \ldots, 1_k$: edge groups or associated subgroups.

Examples

- Baumslag-Solitar groups $\mathrm{BS}_{p,q} = \langle\, a, t \mid t a^p t^{-1} = a^q \,\rangle$
  Conjugacy problem is decidable (actually TC^0-complete).

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- Baumslag's group (aka Baumslag-Gersten group)

$$\begin{aligned}\mathrm{BG}_{1,2} &= \left\langle\, a, b \;\middle|\; (bab^{-1})a(bab^{-1})^{-1} = a^2 \,\right\rangle \\ &= \left\langle\, \mathrm{BS}_{1,2}, b \;\middle|\; bab^{-1} = t \,\right\rangle\end{aligned}$$

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Theorem (Miasnikov, Ushakov, Won 2006)

The word problem of $\mathrm{BG}_{1,2}$ *is in polynomial time.*

Theorem (Beese 2012)

Conjugacy problem of the Baumslag group is decidable in non-elementary time.

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Theorem (Bogopolski, Martino, and Ventura 2010)

There is a group $\mathbb{Z}^4 \rtimes F_k$ with undecidable conjugacy problem.

Action on the Bass-Serre tree

A fgogog G acts naturally on its Bass-Serre tree.

Definition

The elliptic elements of G are those which fix a vertex of the tree.
The hyperbolic elements are those which act without fixed points.

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Consequence

- $\{$ elliptic elements $\} = \bigcup_{g \in G} g(H \cup K)g^{-1}$, or
- $\{$ elliptic elements $\} = \bigcup_{g \in G} gHg^{-1}$.
- $\{$Hyperbolic elements$\} = G \setminus \{$ elliptic elements $\}$.

Solving the conjugacy problem of hyperbolic elements

Lemma (Collins' Lemma)

Let

- $G = H \star_A K$ *or*
- $G = \langle H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \rangle$

Let $v, w \in \Sigma^*$ *be*

- *cyclically Britton-reduced, (no factor* tat^{-1} *or* $t^{-1}bt$ *in* vv *and* ww *for any* $a \in A$ *or* $b \in \varphi(A)$*),*
- *representing hyperbolic group elements.*

Then

$$v \sim w \iff \text{there is a cyclic permutation } w_2 w_1 \text{ of } w = w_1 w_2$$
$$\text{and } a \in A \text{ such that } v = a w_2 w_1 a^{-1}.$$

Consequences

Theorem (Diekert, Miasnikov, W. 2014)

The conjugacy problem of $\mathrm{BG}_{1,2}$ *is decidable in polynomial time for hyperbolic elements.*

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Conjecture (Diekert, Miasnikov, W. 2014)

The conjugacy problem of $\mathrm{BG}_{1,2}$ is non-elementary on average.

Consequences

Observation

Let

$$G = \langle\, H, t_1, \ldots, t_k \mid t_i a t_i^{-1} = \varphi_i(a) \text{ for } a \in A_i,\ i = 1, \ldots, k \,\rangle$$

with A_i finite for all i. If the the word problem of G is decidable, then the conjugacy problem of G is decidable for hyperbolic elements.

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Proof.

Input: v, w
Simply test for all $a \in \bigcup_i A_i$ and all cyclic permutations $w_2 w_1$ of w whether $v = a w_2 w_1 a^{-1}$. □

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The proof relies on:

Theorem (Frumkin 1977, von zur Gathen, Sieveking 1978)

Given a system of linear equation with integer coefficients, it can be determined in polynomial time whether it has an integral solution and, if so, the solution can be computed in polynomial time.

Consequences

Proof

Choose bases for H and for the A_i. This defines integer matrices $M_i^{(1)}, M_i^{(-1)}$ for the inclusions

$$\mathrm{id} : A_i \to H, \qquad \varphi_i : A_i \to H.$$

- Subgroup membership problem for A_i (resp. $\varphi(A_i)$) reduces to a system of linear integer equations.
- Britton reductions $t_i g t_i^{-1} \to \varphi_i(g)$ in polynomial time.
- Compute cyclically Britton-reduced words in polynomial time.

Proof (Cont.)

Apply Collins' Lemma:

- Check all cyclic permutations.
- Let $v = t_{i_1}^{\varepsilon_1} g_1 \cdots t_{i_n}^{\varepsilon_n} g_n, \quad w = t_{i_1}^{\varepsilon_1} h_1 \cdots t_{i_n}^{\varepsilon_n} h_n$
  be cyclically reduced with $g_i, h_i \in H$. Then there is some $a \in \bigcup_i A_i$ with $ava^{-1} =_G w$ iff the system of equations

  $$M_{i_j}^{(\varepsilon_j)} x_j - M_{i_{j+1}}^{(\varepsilon_j)} x_{j+1} + g_j = h_j \qquad \text{for } 1 \le j \le n,$$

  has an integral solution $x_1, \ldots, x_n$.

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A problem $\mathcal{P}$ is in polynomial time (resp. decidable) in a strongly generic setting if there is a partial algorithm $\mathcal{A}$ and a strongly generic set S such that

1. $\mathcal{A}$ solves $\mathcal{P}$ (in polynomial time) on all inputs from S.
2. $\mathcal{A}$ may refuse to give an answer or it might not terminate, but only on inputs outside S.
3. If $\mathcal{A}$ gives an answer, then the answer must be correct.

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The algorithm $\mathcal{A}$ never fools and solves (in polynomial time) correctly "all" random inputs.

Strongly generic algorithms

Theorem (Miasnikov, Rybalov 2008)

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Theorem (Borovik, Miasnikov, Remeslennikov 2005)

The conjugacy problem of Miller's group is strongly generically decidable in polynomial time.

Hyperbolic elements form a strongly generic subset if . . .

Theorem (Main Theorem)

- *Let* $G = H \star_A K$ *be an amalgamated product such that* $[H : A] \geq 3$ *and* $[K : A] \geq 2$*, or let*
- $G = \langle\, H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \,\rangle$ *be an HNN extension with* $[H : A] \geq 2$ *and* $[H : \varphi(A)] \geq 2$*.*

Then the set of words representing hyperbolic elements in G *is strongly generic in* Σ^**.*

Proof: uses the theory of amenable graphs.

Amenability

Notation

Let $\Gamma = (V, E)$ be a locally finite undirected graph.
For $e \in E$ let $\iota(e)$ be its source and $\tau(e)$ its target.

- Γ satisfies the Gromov condition if there exists a map $f : V \to V$ such that $\sup_{v \in V} d(f(v), v) < \infty$ and $\left|f^{-1}(v)\right| \geq 2$ for all $v \in V$ where $d(u, v)$ denotes the distance.

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- Γ satisfies the doubling condition if there exists some $k \in \mathbb{N}$ such that for every finite $U \subseteq V$ we have

$$\left|\left\{ v \in V \;\middle|\; d(v, U) \leq k \right\}\right| \geq 2\,|U|\,.$$

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- A random walk on a (directed) graph starts at some vertex, chooses an outgoing edge uniformly at random and goes to the target vertex, then it chooses the next edge...

Amenability

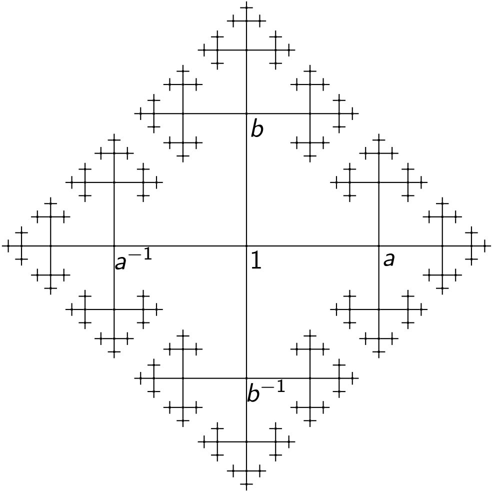
Proposition (Kesten 1959, Gerl 1988, Gromov 1993)

Let $\Gamma = (V, E)$ be a d-regular undirected graph. Then the following statements are equivalent and define amenability:

1. *Γ satisfies the Gromov condition, i. e., there exists a map $f : V \to V$ such that $\sup_{v \in V} d(f(v), v) < \infty$ and $\left|f^{-1}(v)\right| \geq 2$ for all $v \in V$.*
2. *Γ satisfies the doubling condition: there exists some $k \in \mathbb{N}$ such that for every finite $U \subseteq V$ we have*
$$\left|\left\{ v \in V \;\middle|\; d(v, U) \leq k \right\}\right| \geq 2\,|U|\,.$$
3. *The random walk on Γ has exponentially decreasing return probability.*

Examples

The Cayley graph of the free group $F_{\{a,b\}}$ is non-amenable:



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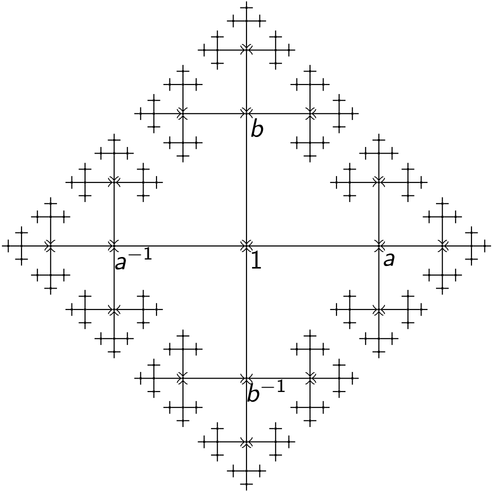

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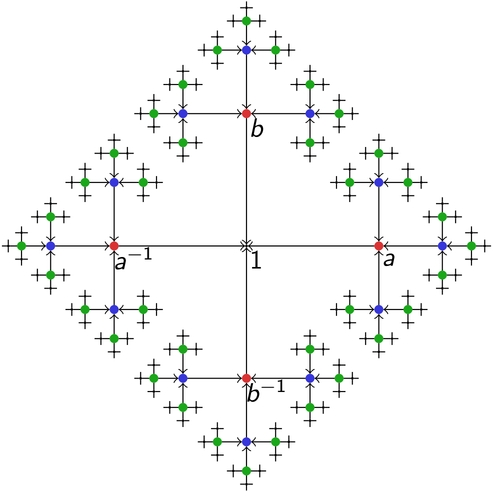

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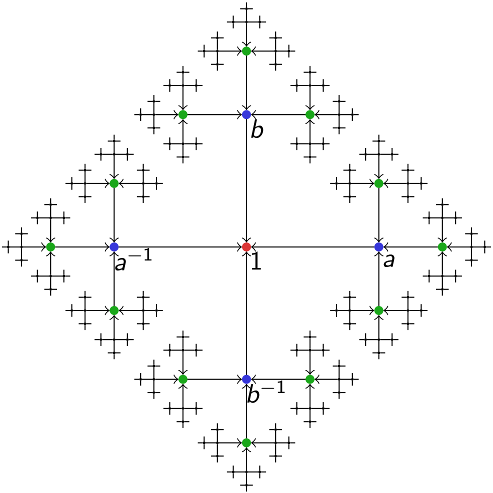

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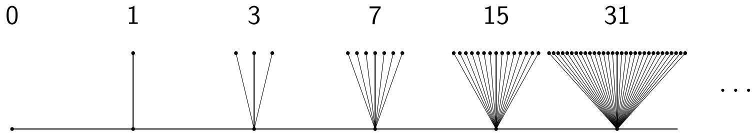
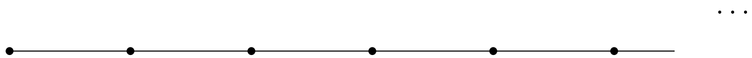
Amenability of locally finite graphs is not a quasi-isometry invariant!!!



The graph above satisfies the Gromov condition, but it is quasi-isometric to an amenable graph.



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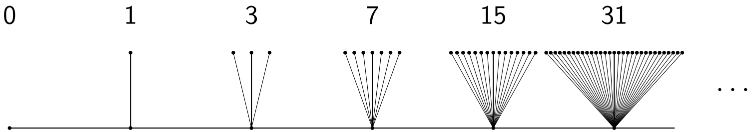

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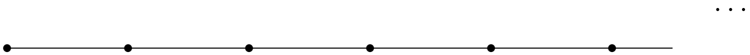

But: for d-regular graphs it is a quasi-isometry invariant.

Schreier graphs

Schreier graph $\Gamma = \Gamma(G, P, \Sigma)$ of G with respect to a subgroup P and set of generators $\Sigma \subseteq G$:

- Vertices: $V(\Gamma) = P \backslash G = \{\, Pg \mid g \in G \,\}$ = right cosets.

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Aim: show non-amenability of Schreier graph.

Characterization theorems

Theorem

Let $G = H \star_A K$ *with* $[H : A] \geq [K : A] \geq 2$ *and* $P \in \{H, K\}$ *and let* $\Sigma = \Sigma^{-1}$ *generate* G*.*
Then the Schreier graph $\Gamma(G, P, \Sigma)$ *is non-amenable iff* $[H : A] \geq 3$*.*

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Theorem

Let $G = \langle\, H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \,\rangle$ *be an HNN extension and let* $\Sigma = \Sigma^{-1}$ *generate* G*.*
Then the Schreier graph $\Gamma(G, H, \Sigma)$ *is non-amenable iff both* $[H : A] \geq 2$ *and* $[H : \varphi(A)] \geq 2$*.*

Example

Let $\mathrm{BS}_{p,q} = \left\langle\, a, t \;\middle|\; t a^p t^{-1} = a^q \,\right\rangle$ be the Baumslag-Solitar group with $1 \leq p \leq q$. Then the Schreier graph $\Gamma(\mathrm{BS}_{p,q}, \langle a \rangle, \{a, \bar{a}, t, \bar{t}\})$ is non-amenable iff $p \neq 1$.

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The Schreier graph $\Gamma(\mathrm{BG}_{1,2}, \mathrm{BS}_{1,2}, \{a, \overline{a}, b, \overline{b}\})$ is non-amenable.

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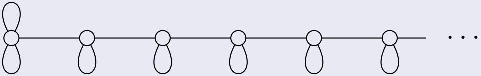
Example

The Schreier graph $\Gamma(H \rtimes F_k, H, \Sigma)$ is non-amenable iff $k \geq 2$.

Proof

Proof for amalgamated products

For the only-if direction we assume $[H : A] = [K : A] = 2$. Then the Schreier graph $\Gamma(G, P, \Sigma)$ is amenable:



Proof

Lemma (Normal forms for amalgamated products)

Fix transversals $C \subseteq H$ and $D \subseteq K$ for cosets of A in H and K with $1 \in C \cap D$ s. t. the decompositions

$$H = AC, \qquad K = AD$$

are unique.

Every group element $g \in G = H \star_A K$ can be uniquely written as

$$g =_G x_0 \cdots x_k$$

for some $k \in \mathbb{N}$, $x_0 \in H \cup K$ such that for all $1 \leq i \leq k$ we have

$$x_i \in C \cup D \setminus \{1\};$$
$$x_{i-1} \in H \iff x_i \in K.$$

Proof

Proof for amalgamated products (Cont.)

Let $[H : A] \geq 3$. We show the Gromov condition (1).
Let $f : P\backslash G \to P\backslash G$ as follows:
Fix $c \neq c' \in C \setminus \{1\}$ and $d \in D \setminus \{1\}$.

- For a normal form $x_0 \cdots x_k$ with $x_k = d$ and $x_{k-1} \in \{c, c'\}$, set $f(Px_0 \cdots x_k) = Px_0 \cdots x_{k-2}$.
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✓ For every normal form w, either wcd and $wc'd$ or wdc and wdc' are normal forms. Hence, $\left|f^{-1}(Pw)\right| \geq 2$ for all $w \in G$.

Back to Conjugacy

Baumslag group

Theorem (Diekert, Miasnikov, W. 2014)

Conjugacy in the Baumslag group $\mathrm{BG}_{1,2}$ *can be solved in polynomial time in a strongly generic setting by some algorithm which always stops and which has non-elementary average time complexity.*

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Hence, there are natural problems where average case complexity is meaningless! Because average case is not better than worst case and the worst case is useless.

Corollary

Let G be a finitely generated group with more than one end. If the word problem of G is decidable in polynomial time, then the conjugacy problem of G is decidable in polynomial time in a strongly generic setting.

Proof.

By Stallings' Structure Theorem, G can be written as amalgamated product or HNN extension over some finite subgroup. □

Free abelian vertex groups

Corollary

- *If one of the following three cases holds*
  - $G = H \star_A K$ *is an amalgamated product with* H, K *f. g. free abelian and* $[H : A] \geq 3$, $[K : A] \geq 2$,
  - $G = \langle H, t \mid tat^{-1} = \varphi(a) \text{ for } a \in A \rangle$ *is an HNN extension with* H *f. g. free abelian and both* $[H : A] \geq 2$ *and* $[H : \varphi(A)] \geq 2$,
  - G *is a fundamental group of a reduced finite graph of groups with f. g. free abelian vertex groups and at least two edges,*

  then the conjugacy problem of G *is decidable in polynomial time on a strongly generic set.*

Application

The conjugacy problem of the $\mathbb{Z}^4 \rtimes F_n$ group with undecidable conjugacy problem (Bogopolski, Martino, Ventura 2010) is strongly generically in polynomial time.

Thank you!