Hardness of equations over finite solvable groups under the exponential time hypothesis

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Equations in groups

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W. l. o. g. of the form

$$\alpha = 1$$

for an expression $\alpha \in (G \cup \mathcal{X} \cup \mathcal{X}^{-1})^*$ (with variables $\mathcal{X}$).

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Constant: The group G

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In many infinite groups these problems are undecidable!

Complexity of equations in groups

In finite groups $\mathrm{EQN\text{-}SAT}(G)$ is in NP:

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- if yes, then α is not an identity.

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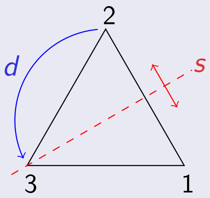
Commutators: $[x, y] = x^{-1}y^{-1}xy = \begin{cases} ?? & \text{if } x \neq 1 \text{ and } y \neq 1 \\ 1 & \text{otherwise.} \end{cases}$

Examples: S_3 and G^*



$$\begin{aligned} S_3 &= \text{group of permutations over three elements} \\ &= \text{symmetry group of a regular triangle} \\ &= \{1, \underbrace{(1\,2)}_{s}, (1\,3), (2\,3), \underbrace{(1\,2\,3)}_{d}, (1\,3\,2)\} \end{aligned}$$

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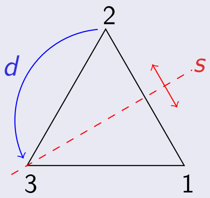

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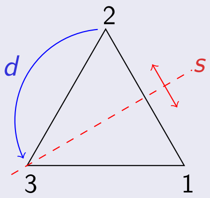

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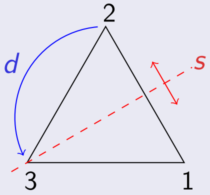

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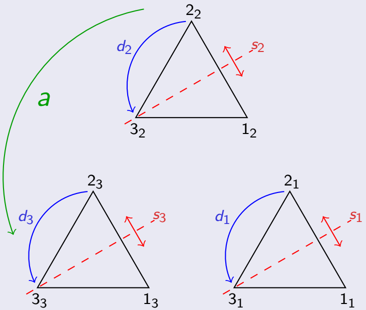

$$G^* = G_{648,705} = (S_3 \times S_3 \times S_3) \rtimes C_3$$

with $a(x, y, z) = (z, x, y)a$

Commutators: $[x, y] = x^{-1}y^{-1}xy$ and $[x_1, \ldots, x_k] = \bigl[[x_1, \ldots, x_{k-1}], x_k\bigr]$

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The Fitting length $\mathrm{FitLen}(G)$ (nilpotent length) of G is the smallest k such that there are normal subgroups

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_k = G$$

with N_i/N_{i-1} nilpotent for all $i = 1, \dots, k$.

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- $(S_3 \times S_3 \times S_3)/(C_3 \times C_3 \times C_3) = (C_2 \times C_2 \times C_2)$
- $G^*/(S_3 \times S_3 \times S_3) = C_3$

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$\rightsquigarrow$ no $2^{o(n+m)}$-time algorithm for $3\mathrm{SAT}$ under ETH.

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Theorem (Idziak, Kawałek, Krzaczkowski, LICS 2020)

$\mathrm{EQN\text{-}SAT}(S_4)$ *and* $\mathrm{EQN\text{-}ID}(S_4)$ *are not in* P *under ETH.*

(S_4 = symmetric group on 4 elements)

C-Coloring

A C-coloring for $C \in \mathbb{N}$ of a graph $\Gamma = (V, E)$ is a map $\chi : V \to [1 \mathinner{..} C]$.
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- NP-complete for $C \geq 3$
- 3-Coloring cannot be solved in time $2^{o(|V|+|E|)}$ unless ETH fails (see e. g. Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, Saurabh, Thm. 14.6).
- $\rightsquigarrow$ for every $C \geq 3$, C-Coloring cannot be solved in time $2^{o(|V|+|E|)}$ unless ETH fails.

Reduce 2-Coloring to EQN-SAT(S_3)

$\Gamma = (V, E)$ graph with $V = \{1, \ldots, n\}$
$E = \{e_1, \ldots, e_m\}$ where $e_k = \{i_k, j_k\}$

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- For every vertex i introduce a variable X_i.

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Length: $|\beta| \approx 2^m$.

$$[d, \alpha_1] = d^{-1}{\alpha_1}^{-1} d\alpha_1$$
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$$[d, \alpha_1, \alpha_2, \alpha_3] = \alpha_2^{-1}{\alpha_1}^{-1} d^{-1}\alpha_1 d\alpha_2 d^{-1}{\alpha_1}^{-1} d\alpha_1\,{\alpha_3}^{-1}\,{\alpha_1}^{-1} d^{-1}\alpha_1 d\alpha_2^{-1} d^{-1}{\alpha_1}^{-1} d\alpha_1\alpha_2\,\alpha_3$$

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Thank you!