The isomorphism problem for finite extensions of free groups is in PSPACE

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Dehn's fundamental problems

Let G be a group, generated by a finite set Σ and $p : \Sigma^* \to G$ the canonical projection. Write $\overline{a}$ for the letter $a^{-1} \in \Sigma$.

- Word problem: Given $w \in \Sigma^*$. Question: Is $w = 1$ in G?
  $$\mathrm{WP}(G) = p^{-1}(1)$$
- Conjugacy problem: Given $v, w \in \Sigma^*$. Question: $v \sim w$?
  $$(\exists\, z \in G \text{ such that } zvz^{-1} = w?)$$
- Isomorphism problem: Are the groups $\langle \Sigma \mid R \rangle$ and $\langle \Sigma' \mid R' \rangle$ isomorphic?

Free groups

The free group with basis X (where $\Delta = X \cup \overline{X}$):

$$F(X) = \Delta^* \Big/ \{ \, a\bar{a} = \bar{a}a = 1 \mid a \in X \, \}$$

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PDA for $\mathrm{WP}(F(X))$: when reading $a \in \Delta$:

```
**if** stack-top $\neq \overline{a}$ **then**
    push($a$);
**else**
    pop;
**endif**
```

Virtually free groups

Virtually free = free subgroup of finite index

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Virtually free presentation:

- generating set X of F,
- a system of representatives $R \subseteq G$ of Q ($\rightsquigarrow G = F \cdot R$)
- multiplication rules: for $q \in R, a \in R \cup X$ there are $f \in F, r \in R$ with

$$qa = fr.$$

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Every element can be written as $x^k a^\varepsilon$ with $k \in \mathbb{Z}$, $\varepsilon \in \{ 0, 1 \}$.

Virtually free groups are context-free

PDA: given a word $w \in (X \cup \overline{X} \cup R)^*$, rewrite it as fr with $f \in F, r \in R$, keep f on the stack, r in the state.

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Theorem (Muller, Schupp, 1983)

A group is finitely generated virtually free iff it is context-free (the word problem is a context-free language).

The isomorphism problem for virtually free groups

$$1 \to F_i \to G_i \to Q_i \to 1$$

G_1

$$F_1 = \mathbb{Z} = \langle x \rangle, \qquad Q_1 = \mathbb{Z}/2\mathbb{Z}, \qquad R_1 = \{ 1, a \}$$

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G_2

$$F_2 = \mathbb{Z} = \langle y \rangle\,, \qquad Q_2 = \mathbb{Z}/2\mathbb{Z}, \qquad R_2 = \{\, 1, b \,\}$$

with rules $by = yb, \qquad bb = y,$

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G_3

$$F_3 = \mathbb{Z} = \langle z \rangle\,, \qquad Q_3 = \mathbb{Z}/2\mathbb{Z}, \qquad R_3 = \{\, 1, c \,\}$$

with rules $cz = zc, \qquad cc = zz.$

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$$F_2 = \mathbb{Z} = \langle y \rangle, \qquad Q_2 = \mathbb{Z}/2\mathbb{Z}, \qquad R_2 = \{1, b\}$$

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G_3

$$F_3 = \mathbb{Z} = \langle z \rangle, \qquad Q_3 = \mathbb{Z}/2\mathbb{Z}, \qquad R_3 = \{1, c\}$$

with rules $cz = zc, \qquad cc = zz.$

Then $G_1 \cong G_3 \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (via $z \mapsto x$, $c \mapsto ax$) and $G_2 \cong \mathbb{Z}$.

The isomorphism problem for virtually free groups

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The isomorphism problem for virtually free groups is decidable (input: presentations with the promise to be virtually free).

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Theorem (Sénizergues, 1996)

The isomorphism problem for context-free groups is primitive recursive (input: pda or c.f. grammar).

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Theorem (Sénizergues, 1996)

The isomorphism problem for context-free groups is primitive recursive (input: pda or c.f. grammar).

Theorem (S.,W.)

The isomorphism problem for virtually free groups is in PSPACE, for context-free groups it is in $\mathrm{DSPACE}(2^{2^{\mathcal{O}(n)}})$.

Definition (Graph of Groups)

A graph of groups $\mathcal{G}$ is a connected graph $Y = (V(Y), E(Y))$ and

1. for each vertex $P \in V(Y)$, a vertex group G_P,
2. for each edge $y \in E(Y)$, an edge group G_y such that $G_y = G_{\overline{y}}$.
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Example

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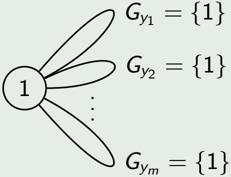

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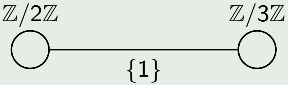

$$\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

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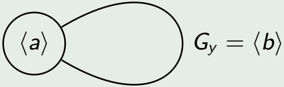

$\mathbf{BS}_{p,q} = \left\langle\, a, y \;\middle|\; ya^p y^{-1} = a^q \,\right\rangle$ with embeddings $b \mapsto a^p$ and $b \mapsto a^q$

Theorem (Karrass, Pietrowski, Solitar 73)

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Theorem (Guirardel, Levitt 07; Clay, Forester 09)

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be reduced finite graph of groups with finite vertex groups. Then $\pi_1(\mathcal{G}_1, P_1) \cong \pi_1(\mathcal{G}_2, P_2)$ iff $\mathcal{G}_1$ can be transformed into $\mathcal{G}_2$ by a sequence of slide moves.

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Example (Slide move)

If there is some $g \in G_P$ such that $g^{-1} G_x^x g \leq G_y^y$.

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Corollary

It can be decided in NSPACE(n) *whether $\pi_1(\mathcal{G}_1, P_1) \cong \pi_1(\mathcal{G}_2, P_2)$ given two graph of groups $\mathcal{G}_1$ and $\mathcal{G}_2$ with finite vertex groups.*

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Krstić's proof.

- For both input groups guess a gog + an isomorphism
- verify that the guesses are correct
- check the two gogs for isomorphism $\square$

New approach

Show that the gog and the isomorphism are "small".

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Theorem (S.,W.)

The following problem is in $\mathrm{NTIME}(2^{2^{\mathcal{O}(n)}})$*:*
Input: a c.f grammar for the word problem of a group G*,*
Compute a gog $\mathcal{G}$ *with finite vertex groups and* $\pi_1(\mathcal{G}, P) \cong G$

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Theorem (S.,W.)

The following problem is in NP*:*
Input: a group G given as virtually free presentation,
Compute a gog $\mathcal{G}$ with finite vertex groups and $\pi_1(\mathcal{G}, P) \cong G$.

Small graph of groups

Main Lemma

Let G be given as context-free grammar of size $N \geq 4$ for its word problem. There is a graph of groups $\mathcal{G}$ over Y and an isomorphism $\varphi : \pi_1(\mathcal{G}, T) \to G$ with

1. $|V(Y)| \leq N^{50 \cdot 2^N}$,
2. $|G_P| \leq N^{50 \cdot 2^N}$ for all $P \in V(Y)$,
3. $|\varphi(a)| \leq 24 \cdot N^{175 \cdot 2^N}$ for every $a \in \Delta =$ generating set of $\pi_1(\mathcal{G}, T)$.

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If G is given as virtually free presentation of size $M \geq 4$, then

1. $|V(Y)| \leq M + 1$,
2. $|G_P| \leq M$ for all $P \in V(Y)$,
3. $|\varphi(a)| \leq 12(M+1)^6$ for every $a \in \Delta$.

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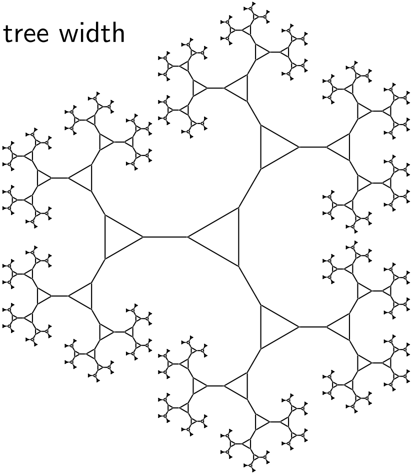
$\rightsquigarrow M^{\mathcal{O}(1)}$

Virtually free groups are "tree-like"

Let $\Gamma(G)$ be the Cayley graph of a c.f. group G. Then:

- $\Gamma(G)$ is quasi-isometric to a tree
- $\Gamma(G)$ has finite tree width



The Cayley graph of $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ has finite tree-width.

Virtually free groups are “tree-like”

Let $\Gamma(G)$ be the Cayley graph of a c.f. group G. Then:

- $\Gamma(G)$ is quasi-isometric to a tree
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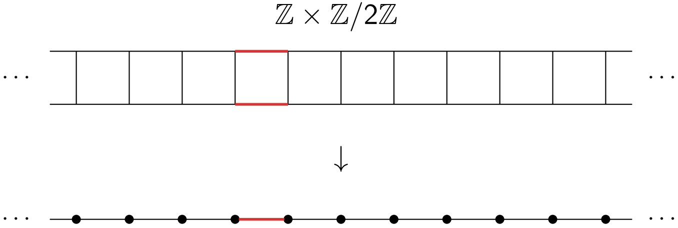

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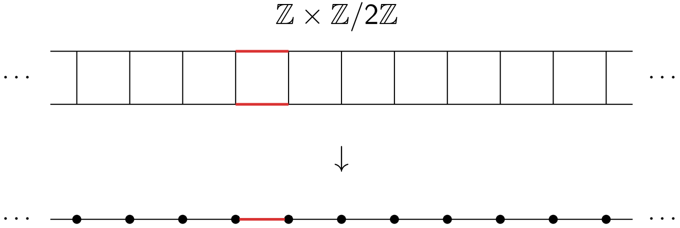

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Key point for the main Lemma: bound size of cuts and "vertices"

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Thank you!