On the dimension of matrix embeddings of torsion-free nilpotent groups

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τ-Groups

Definition

τ-group $=$ finitely generated torsion-free nilpotent group.

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Examples:

- unitriangular matrices $UT_n(\mathbb{Z})$
  (upper triangular and diagonal entries 1)
- Heisenberg groups
- free nilpotent groups
  $F_{k,c} = \langle a_1, \ldots, a_k \mid [x_1, \ldots, x_{c+1}] = 1 \text{ for } x_1, \ldots, x_{c+1} \in F_{k,c} \rangle$
  where ($[x_1, \ldots, x_{c+1}] = [[x_1, \ldots, x_c], x_{c+1}]$)
- $\langle a, b, c, d, e \mid [a, b] = [b, c] = d^2 e,\ [a, c] = e^3,$
  $[e, x] = [d, x] = 1\ \forall x \rangle$

Embeddings of τ-Groups

Theorem (Jennings 1955)

Every τ-group can be embedded into $UT_N(\mathbb{Z})$ for some $N \in \mathbb{N}$.

The embedding is given by the G-action on $\mathbb{Q}G/I^{c+1}$ where $I = \left\{ \sum_g \alpha_g g \;\middle|\; \sum_g \alpha_g = 0 \right\}$ is the augmentation ideal.

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Nickels seems to be the "best" for doing actual computations.

Embeddings of τ-Groups

Why embeddings into matrices are useful:

- lot known about matrices – linear algebra
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Desired properties properties of embeddings:

- small dimension (little overhead when doing computations)
- easy to compute
- undistorted (geometry is preserved)
- preserves conjugacy etc.

Mal'cev coordinates

Let G be a τ-group with Mal'cev basis $(a_1, \ldots, a_n) = \vec{a}$.

- Each $g \in G$ has a unique normal form

$$g = a_1^{x_1} \cdots a_n^{x_n} =: \vec{a}^{\vec{x}}$$

with $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ and such that

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Example

$F_{2,2} = \langle a_1, a_2 \mid [[x, y], z] = 1 \text{ for } x, y, z \in F_{2,2} \rangle$

- (a_1, a_2) is not a Mal'cev basis since $a_2 a_1$ cannot be written as $a_1^x a_2^y$

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- $F_{2,2} = UT_3(\mathbb{Z}) = H_3 = \langle a_1, a_2, a_3 \mid [a_2, a_1] = a_3,\ [a_3, a_1] = [a_3, a_2] = 1 \rangle$

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Theorem (P. Hall, 1957)

$$q_1, \ldots, q_n \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

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$$UT_N(\mathbb{Z}) \leq \mathrm{Aut}(\mathbb{Q}^N)$$

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$$\begin{aligned}(\mathbb{Q}G)^* &= \{f : \mathbb{Q}G \to \mathbb{Q} \mid f \text{ is linear}\} \\ &= \{f : G \to \mathbb{Q}\} = \{f : \mathbb{Z}^n \to \mathbb{Q}\}\end{aligned}$$

is a G-module:

$$f^g(z) = f(z \cdot g^{-1}) \qquad \text{for } g \in G,\ f \in (\mathbb{Q}G)^* \text{ and } z \in \mathbb{Q}G$$

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$\rightsquigarrow$ compute f^g = substitute multiplication polys into f.

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Lemma (Nickel, 2006)

The submodule $M = \operatorname{span}\left(\{\, t_1, \ldots, t_n \,\}^G\right)$ of $(\mathbb{Q}G)^*$ generated by $t_1, \ldots, t_n$ is a finite dimensional faithful G-module. Moreover, it has a basis such that the corresponding matrices are of unitriangular shape.

How to Compute the Embedding

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- Extend $\{\, t_1, \ldots, t_n \,\}$ to a $\mathbb{Q}$-basis B of $\mathrm{span}\{\, t_1, \ldots, t_n \,\}^{a_1^{\mathbb{Z}}}$ (finite dimensional):
  - Compute polynomials $q_1^{(1)}, \ldots, q_n^{(1)}$ with $a_1^{x_1} \cdots a_n^{x_n} \cdot a_1^{-1} = a_1^{q_1^{(1)}} \cdots a_n^{q_n^{(1)}}$
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  - substitute them into the polynomials of the previous step until no new linearly independent polynomials appear.
- Extend B to a $\mathbb{Q}$-basis of $\operatorname{span}(B^{a_2{}^{\mathbb{Z}}})$
- ...

Example: 3-dim Heisenberg group

$$H_3 = UT_3(\mathbb{Z}) = F_{2,2} = \langle a_1, a_2, a_3 \mid [a_1, a_3] = [a_2, a_3] = 1,\ [a_1, a_2] = a_3 \rangle$$

$$a_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$t_i^{a_1}(a_1^{x_1} a_2^{x_2} a_3^{x_3}) = \begin{cases} x_1 - 1 & = t_1(a_1^{x_1} a_2^{x_2} a_3^{x_3}) - 1, \\ x_2 & = t_2(a_1^{x_1} a_2^{x_2} a_3^{x_3}), \\ x_3 + x_2 & = t_3(a_1^{x_1} a_2^{x_2} a_3^{x_3}) + t_2(a_1^{x_1} a_2^{x_2} a_3^{x_3}), \end{cases}$$

Example: 3-dim Heisenberg group

Similarly,

$$
\begin{aligned}
{t_1}^{a_2^k}(a_1^{x_1}a_2^{x_2}a_3^{x_3}) &= x_1 = t_1 \\
{t_2}^{a_2^k}(a_1^{x_1}a_2^{x_2}a_3^{x_3}) &= x_2 - k = t_2 - k \cdot 1 \\
1^{a_2^k}(a_1^{x_1}a_2^{x_2}a_3^{x_3}) &= 1 && \text{(constant polynomial)} \\
{t_3}^{a_2^k}(a_1^{x_1}a_2^{x_2}a_3^{x_3}) &= x_3 = t_3 \\
{t_1}^{a_3^k}(a_1^{x_1}a_2^{x_2}a_3^{x_3}) &= x_1 = t_1 \\
{t_2}^{a_3^k}(a_1^{x_1}a_2^{x_2}a_3^{x_3}) &= x_2 = t_2 \\
{t_3}^{a_3^k}(a_1^{x_1}a_2^{x_2}a_3^{x_3}) &= x_3 - 1 = t_3 - k \cdot 1
\end{aligned}
$$

$\rightsquigarrow$ $(t_3, t_2, t_1, 1)$ is a $\mathbb{Q}$-basis for the H-submodule. So H can be embedded into $UT_4(\mathbb{Z})$.

Example: 3-dim Heisenberg group

With the basis $(t_3, t_2, t_1, 1)$:

$$a_1 \mapsto \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad a_2 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a_3 \mapsto \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: 3-dim Heisenberg group

For comparison: Jennings' embedding of H has dimension 7.

$$a_1 \mapsto \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad a_2 \mapsto \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

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Heisenberg groups

$(2m+1)$-dimensional Heisenberg group with Mal'cev basis $(a_1, \dots, a_{2m+1})$

$$H = \big\langle a_1, \dots, a_{2m+1} \;\big|\; [a_i, a_{m+i}] = a_{2m+1} \text{ for } 1 \le i \le m, \\ [a_i, a_j] = 1 \text{ for } i = 2m+1 \text{ or } |i-j| \ne m \big\rangle$$

$$H = \begin{pmatrix} 1 & \star & \star & \cdots & \star & \star \\ & 1 & 0 & \cdots & 0 & \star \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & 1 & 0 & \star \\ & 0 & & & 1 & \star \\ & & & & & 1 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \dots, a_m = \begin{pmatrix} 1 & 0 \cdots 0 & 1 & 0 \\ & 1 & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}$$

Heisenberg groups

Theorem

For the $(2m+1)$-dimensional Heisenberg group

- *Jennings' embedding has size* $2m^2 + 3m + 2$,
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Heisenberg groups

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Proof.

For $1 \leq j \leq m$, we have

$$t_i^{a_j^{-k}}(\vec{a}^{\vec{x}}) = \begin{cases} x_j - k & \text{for } i = j \\ x_i & \text{for } i \neq j \text{ and } i \neq 2m+1 \\ x_{2m+1} + kx_{m+j} & \text{for } i = 2m+1 \end{cases}$$

For $m + 1 \leq j \leq 2m + 1$,

$$t_i^{a_j^{-k}}(\vec{a}^{\vec{x}}) = \begin{cases} x_j - k & \text{for } i = j \\ x_i & \text{for } i \neq j \end{cases}$$

Size (dimension) of embeddings

Embed a τ-group G into $UT_N(\mathbb{Z})$.

Trivial lower bound for arbitrary embeddings:

$$\frac{N(N-1)}{2} = \text{Hirsch-length}(UT_N(\mathbb{Z})) \geq \text{Hirsch-length}(G)$$

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Nickel's experiments (2006) for embedding $UT_m(\mathbb{Z})$ into $UT_N(\mathbb{Z})$

m	2	3	4	5	6	7	8	9
Hirsch-length	1	3	6	10	15	21	28	36
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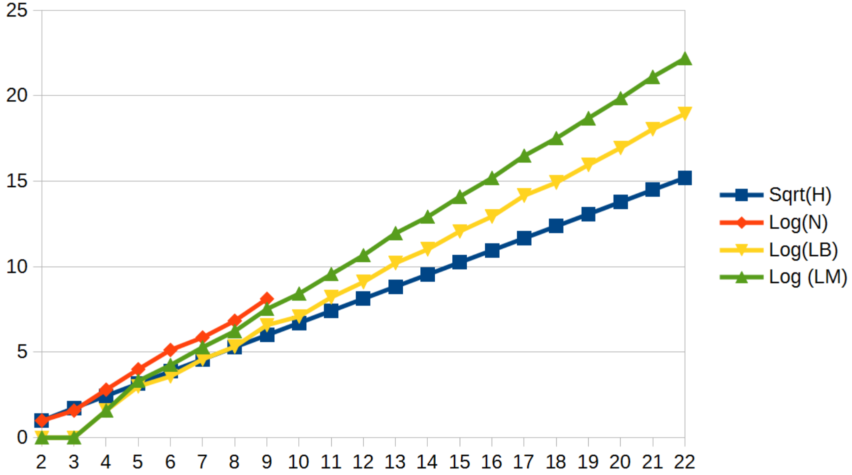
2^{m-1}	2	4	8	16	32	64	128	256

Size (dimension) of embeddings



$$s_{i,j} = \begin{array}{cc} \begin{matrix} & & & \downarrow i & \end{matrix} & \\ \begin{pmatrix} 1 & & & & \\ & 1 & & 1 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix} & \begin{matrix} \\ \leftarrow j \\ \\ \\ \\ \end{matrix} \end{array}$$

$\{ s_{i,j} \mid 1 \leq i < j \leq m \}$ is a Mal'cev basis (if properly ordered).

Mal'cev bases for $UT_m(\mathbb{Z})$

Let $A = (a_1, \ldots, a_n)$ with $n = \frac{m(m-1)}{2}$ be the Mal'cev with

$$a_1 = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & 1 & 0 \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \ldots$$

$$a_i = \begin{pmatrix} 1 & & & & \\ & 1 & & 1 & \\ & & 1 & & \\ & & & 1 & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

the i-th matrix in this order

Let $B = (b_1, \ldots, b_n)$ with $n = \frac{m(m-1)}{2}$ be the Mal'cev with

$$b_1 = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \; b_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & 1 & 0 \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \; b_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \cdots & 0 \\ & 1 & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \ldots$$

$$b_i = \begin{pmatrix} 1 & \uparrow & \uparrow & \uparrow & & \\ & 1 & | & | & 1 & \\ & & 1 & | & \uparrow & \\ & & & 1 & | & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

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Theorem

Nickel's embedding of $UT_m(\mathbb{Z})$ with Mal'cev basis A into $UT_N(\mathbb{Z})$ satisfies

$$N \geq 2^{\lfloor \frac{m}{2} \rfloor - 1}.$$

Theorem

Nickel's embedding of $UT_m(\mathbb{Z})$ with Mal'cev basis B into $UT_{N'}(\mathbb{Z})$ satisfies

$$N' = \frac{m(m-1)}{2} + 1.$$

Results on $UT_m(\mathbb{Z})$

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Theorem

Nickel's embedding of $UT_m(\mathbb{Z})$ with Mal'cev basis B into $UT_{N'}(\mathbb{Z})$ satisfies
$$N' = \frac{m(m-1)}{2} + 1.$$

Let $(a_1, \ldots, a_n)$ any ordering of the standard Mal'cev basis $\{ s_{i,j} \mid 1 \leq i < j \leq m \}$ of $UT_m(\mathbb{Z})$.

Theorem

Nickel's embedding of $UT_m(\mathbb{Z})$ into $UT_N(\mathbb{Z})$ satisfies $N \leq 3^m$.

Proof Idea: Lower Bound

Compute
$$t_n^{a_1} = t_n^{s_{1,2}} = \prod_{i=2}^{m-1} x_i + P$$

by applying the commutation rules

$$s_{i,j}^{x}\, s_{k,\ell}^{y} = \begin{cases} s_{k,\ell}^{y}\, s_{i,j}^{x} & \text{if } i \neq \ell \text{ and } j \neq k, \\ s_{k,\ell}^{y}\, s_{i,j}^{x}\ s_{i,\ell}^{xy} & \text{if } j = k, \\ s_{k,\ell}^{y}\, s_{i,j}^{x}\ s_{k,j}^{-xy} & \text{if } i = \ell. \end{cases}$$

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Write $x_{i,j}$ for x_k if $s_{i,j} = a_k$

Recall: $s_{i,j} = \begin{pmatrix} 1 & & & & \\ & 1 & & 1 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix}$

$$s_{1,2}^{x_{1,2}} \cdots s_{m-1,m}^{x_{m-1,m}} s_{1,3}^{x_{1,3}} \cdots s_{2,m}^{x_{2,m}} s_{1,m}^{x_{1,m}} \cdot s_{1,2}^{-1}$$

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$$= s_{1,2}^{x_{1,2}}\, s_{2,3}^{x_{2,3}} \cdot s_{1,2}^{-1} \cdot s_{3,4}^{x_{3,4}} \cdots s_{m-1,m}^{x_{m-1,m}}\, s_{1,3}^{\star} \cdots s_{2,m}^{\star}\, s_{1,m}^{\star}$$

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Compute
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by applying the commutation rules

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Let G be of nilpotency class c and $k = \mathrm{rk}(G/[G,G])$. Then Nickel's embedding has dimension at most

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Theorem (Lo, Ostheimer, 1999)

Jennings' embedding of $F_{k,c}$ has dimension exactly $\sum_{i=0}^{c} k^i$.

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Direct Products

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Proposition

Let M (resp. N) be the dimension of Nickel's embedding of G (resp. H) into $UT_M(\mathbb{Z})$ (resp. $UT_N(\mathbb{Z})$). Then Nickel's embedding of $G \times H$ has dimension

$$M + N - 1.$$

Direct Products

Example

- $G = \mathbb{Z}^k$
- $H = \mathbb{Z}^c \rtimes_\varphi \mathbb{Z}$ where the action $\mathbb{Z}$ on $\mathbb{Z}^c$ is defined by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & 1 & \ddots & \vdots \\ & & 1 & \ddots & 0 \\ & 0 & & \ddots & 1 \\ & & & & 1 \end{pmatrix}$$

Jennings' embedding has the following sizes

- for G: $k+1$
- for H: $2^{\mathcal{O}(\sqrt{c})}$
- for $G \times H$: greater than $\binom{k+c}{c}$ (for $k = c$ this is $\approx 4^k/\sqrt{k}$).

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Open Questions

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Thank you!