The power word problem in free groups

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[1] Joint work with Markus Lohrey

Dehn's fundamental problems and others

Let G be a f. g. group, generated by a finite set $\Sigma = \Sigma^{-1} \subseteq G$.

- Word problem (WP): Given $w \in \Sigma^*$. Question: Is $w = 1$ in G?
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$$\begin{pmatrix} -499 & 5000 \\ -50 & 501 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{10} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{50} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-10}$$

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The power word problem helps

- to solve the knapsack problem in RAAGS (Lohrey, Zetsche, 15), ...
- to understand the compressed word problem better:
  - lower bounds
  - better upper bounds in the special case.

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- We cannot be faster than linear time on one processor, but we can on many processors.
- Parallel computing is more and more important in the "real world".

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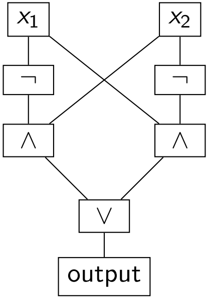

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size = number of gates
depth = longest path from input to output gate
fan-in = number of input-wires per gate

NC = problems which can be solved by a family of circuits of polynomial size and polylogarithmic depth and bounded fan-in.

Inside NC:

- NC^i = solved by a family of circuits of depth $\mathcal{O}(\log^i n)$ and polynomial size with bounded fan-in (= in-degree) $\neg$, $\wedge$, $\vee$ gates.

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Inside NC^1:

- AC^0 = solved by a family of circuits of constant depth and polynomial size with unbounded fan-in $\neg$, $\wedge$, $\vee$ gates.
- TC^0 allows additionally majority gates: $\mathrm{Maj}(w) = 1$ iff $|w|_1 \geq |w|_0$ for $w \in \{0, 1\}^*$.

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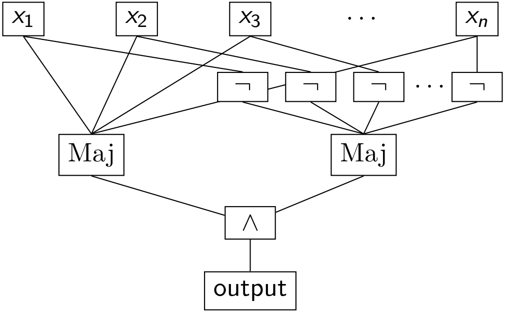

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Theorem (Myasnikov, W. 2017, Lohrey, W.)

If G is f.g. nilpotent or $G = H \wr \mathbb{Z}$ for H f.g. abelian, then $\textsc{PowerWP}(G)$ *is in* TC^0.

Reductions

- For a formal language $L \subseteq \{0, 1\}^*$, $\mathsf{AC}^0(L)$ allows additionally oracle gates for L.
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  $\rightsquigarrow$ Majority is TC^0-complete.

- $\mathsf{TC}^0 = \mathsf{AC}^0(\mathrm{WP}(\mathbb{Z})) \subseteq \mathsf{AC}^0(\mathrm{WP}(F_2))$
- $\mathsf{AC}^0(\mathrm{WP}(F_2)) \subseteq \mathsf{LOGSPACE}$

Word problem of free groups

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Theorem (Lohrey, W.)

The power word problem for free groups is in $\mathrm{AC}^0(\mathrm{WP}(F_2))$.

Overview: small circuit classes

AC^0	$\mathbb{Z}/n\mathbb{Z}$ with one monoid generator
TC^0	$\mathbb{Z}$, linear solvable, free solvable $\textsc{PowerWP}(\mathsf{Ab} \wr \mathbb{Z})$, $\textsc{PowerWP}(\text{nilpotent})$
$\mathsf{NC}^1 = \mathsf{AC}^0(\mathrm{WP}(A_5))$	finite non-solvable, regular languages

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P polynomial time	compressed word problem of free groups,...

Open Questions I

- Is there a natural (non-group theoretic) problem which is $\mathsf{AC}^0(\mathrm{WP}(F_2))$-complete?
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Or even more challenging:

- Separation results: $\mathsf{TC}^0 \neq \mathsf{NC}^1$? $\mathsf{AC}^0(\mathrm{WP}(F_2)) \neq \mathsf{NC}^1$?...
- Can a non-solvable group have word problem in TC^0?

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Three steps:

- Preprocessing
- Make exponents small
- Solve regular word problem

Examples: Power word problem in free groups

Let $F = F(\{a, b\})$ be the free group. Write $\overline{a}$ for a^{-1}

Example 1

$$(a\,b)^{1000}\,a\,b^{-100}\,b^{100}\,a\,b^{-100}\,b^{100}\,\overline{a}\,\overline{a}\,(a\,b)^{-1000}$$

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Example 4

$$(b\,a\,a\,\overline{a}\,b\,a)^{500}\,(b)^{2}\,(\overline{b}\,\overline{b}\,\overline{a}b)^{999}\,(\overline{b}\,\overline{a}\,\overline{b}\,\overline{b}\,a\,b)^{1}(a\,b)^{-1}$$

Preprocessing

$\Omega \subseteq \Sigma^+$ is set of non-empty words p with

(1) p is cyclically reduced,
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$$\Omega = \{ a, b, ab, a\overline{b}, aab, aa\overline{b}, \dots \}$$

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Lemma

Let $p, q \in \Omega$ and v a factor of p^x and w a factor of q^y.
If $vw = 1$ in F and $|v| = |w| \geq |p| + |q| - 1$, then $p = q$.

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Let $p, q \in \Omega$ and v a factor of p^x and w a factor of q^y.
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- By (1), $v = w^{-1}$ as words. $\rightsquigarrow$ v has periods $|p|$ and $|q|$.
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- By (2), $|p| = |q|$.
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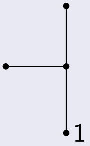

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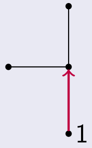

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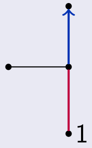

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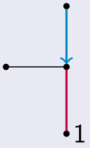

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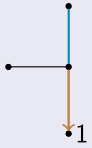

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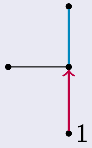

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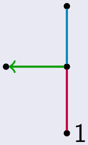

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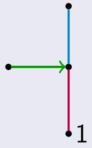

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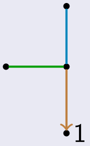
$w_{4,8} = 1$



Computing freely reduced words

Proposition (W., 2016)

Freely reduced words can be computed in $\mathrm{AC}^0(\mathrm{WP}(F))$.

Proof.

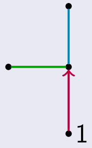
Input: $w = w_1 \cdots w_n$ with $w_i \in \Sigma \cup \Sigma^{-1}$. Set $w_{i,j} = w_{i+1} \cdots w_j$.
Define an equivalence relation $\approx \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ by

$$i \approx j \iff w_i = w_j \text{ and } \begin{cases} w_{i,j} =_F 1 & \text{if } i < j, \\ w_{j,i} =_F 1 & \text{if } j < i. \end{cases}$$

$\rightsquigarrow$ $i \approx j$ iff w_i and w_j are the same edge in the Cayley graph

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ b & b & \overline{b} & \overline{b} & b & \overline{a} & a & \overline{b} & b \end{array}$$

$w_{1,9} = w_{5,9} = 1$



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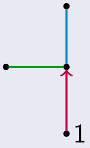

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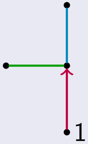

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Can be checked in $\mathsf{AC}^0(\mathrm{WP}(F))$ for all pairs i, j whether $i \approx j$.

Proof. (Contd.)

Define a partial map

$$\overline{\cdot} : \{1, \ldots, n\}/\!\approx \;\to \{1, \ldots, n\}/\!\approx$$

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Computing freely reduced words

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- if $|[i]| > |\overline{[i]}|$, after any sequence of free reductions, there remains one letter w_j for some $j \in [i]$.

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Output all w_j with $j = \max [i]$ for some i with $|[i]| > |\overline{[i]}|$ and delete the other letters.

Make exponents small

Now we have a "nice" instance

$$w = s_0 p_1^{x_1} s_1 \cdots p_n^{x_n} s_n \qquad \text{with } p_i \in \Omega \text{ and } s_i \text{ freely reduced.}$$

We know that

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Nor down to 1:

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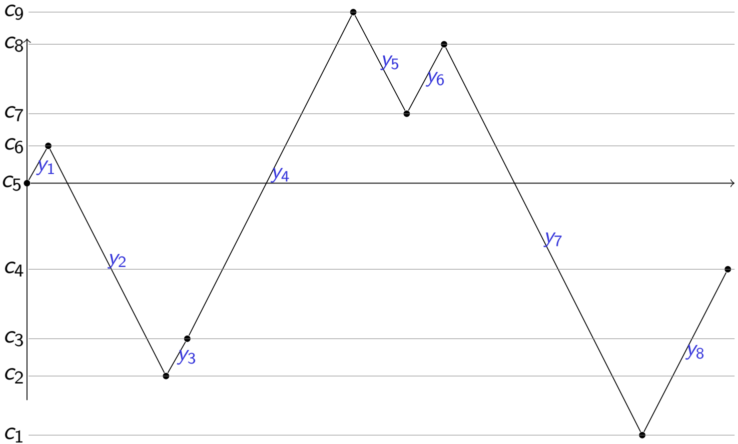

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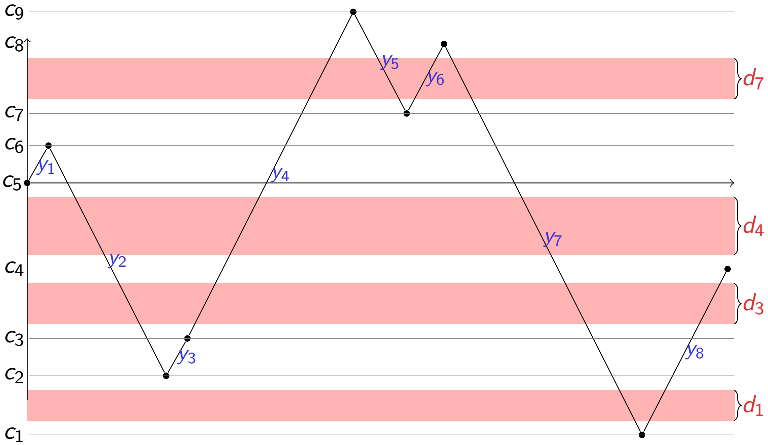

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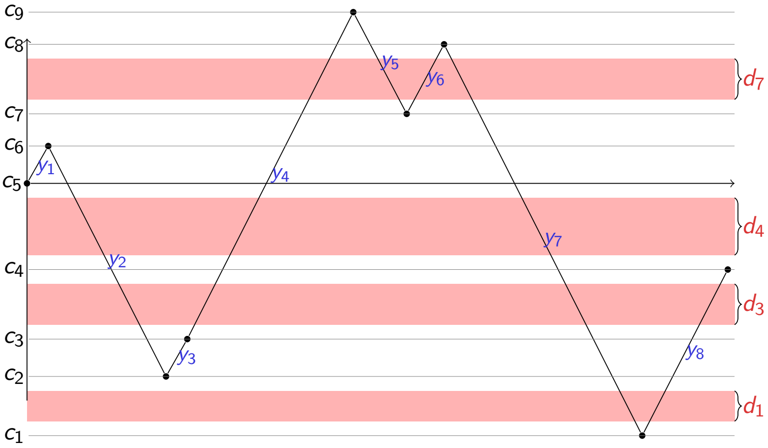

Define $\mathcal{S}(w) = u_0 p^{z_1} u_1 \cdots p^{z_m} u_m$ where $z_i = y_i - \mathrm{sign}(y_i) \cdot \sum_{j \in C_i} d_j$

Make exponents small

Proposition

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Proof of the main theorem.

- Preprocessing gives a "nice word" $w = s_0 p_1^{x_1} s_1 \cdots p_n^{x_n} s_n$.
- For all $p \in \Omega$ which appear in w, compute $\mathcal{S}(w)$ in parallel (iterated addition $\rightsquigarrow$ in TC^0).
- Yields a word of polynomial length $\rightsquigarrow$ apply the ordinary word problem.

Further results on the power word problem

Theorem (Lohrey, W.)

Let G be f.g. and $H \leq G$ of finite index. Then $\textsc{PowerWP}(G)$ *is* NC^1*-many-one-reducible to* $\textsc{PowerWP}(H)$.

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Corollary

The power word problem of f.g. virtually free groups is in $\mathsf{AC}^0(\mathrm{WP}(F_2))$.

Theorem (Lohrey, W.)

Let G be either

- *finite non-solvable*
- *f.g. free of rank ≥ 2.*

Then $\textsc{PowerWP}(G \wr \mathbb{Z})$ *is* coNP*-complete.*

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- $Y_i \mapsto (a \underbrace{t \cdots t}_{p_i})^{M_i} t^{-M} = \underbrace{(a, \underbrace{1, \ldots, 1}_{p_i - 1}, \ldots, a, \underbrace{1, \ldots, 1}_{p_i - 1})}_{M_i \text{ times}}$

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Proof: coNP hardness

- $F_2 \wr \mathbb{Z} = \langle a, b, t \rangle$.
- For any assignment $\sigma : \{X_1, \ldots, X_m\} \to \{0, 1\}$

$$\sigma(F) = 0 \iff \sigma'(w_F) =_{F_2} 1$$

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- By the Chinese Remainder Theorem, this tests all valuations.

Open Questions II

The proof for free groups should be generalizable to

- RAAGs (= graph groups),
- graph products,
- hyperbolic groups,
- HNN extensions and amalgamated products over finite subgroups.

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Example

Let $p = qa$ with $[q, a] = 1$, then q^x is a factor of p^x and cancels with q^{-x} but $p \neq q$!

$\rightsquigarrow$ need more restrictions on Ω

- What if we allow nested exponents:

$$\left(b^{13}\overline{a}\ \left((b\,a^{8}a)^{13}a^{-26}b^{-13}\right)^{12}\right)^{16}\left((\overline{b}\,\overline{a})^{13}a^{13}\right)^{20}$$

  - Conjecture: for constant nesting depth in $\mathrm{AC}^0(\mathrm{WP}(F_2))$ (same approach).
  - Not clear what happens for unbounded nesting depth:
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