The Conjugacy Problem in Free Solvable Groups and Wreath Products of Abelian Groups is in TC^0

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Dehn's fundamental problems and others

Let G be a f. g. group, generated by a finite set $\Sigma = \Sigma^{-1} \subseteq G$.

- Word problem $\mathrm{WP}(G)$: Given $w \in \Sigma^*$. Question: Is $w =_G 1$?
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- The power problem $\mathrm{PP}(G)$: Given $v, w \in \Sigma^*$.
  Question: is $w \in \langle v \rangle$? If yes, compute $k \in \mathbb{Z}$ such that $v^k =_G w$.

Circuit Complexity

Circuit = directed acyclic graph where each vertex is either:

- input gates (has only outgoing edges)
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$$AC^0 \subsetneq TC^0 \subseteq NC^1 \subseteq LOGSPACE \subseteq NC^2 \subseteq \cdots \subseteq NC \subseteq P$$

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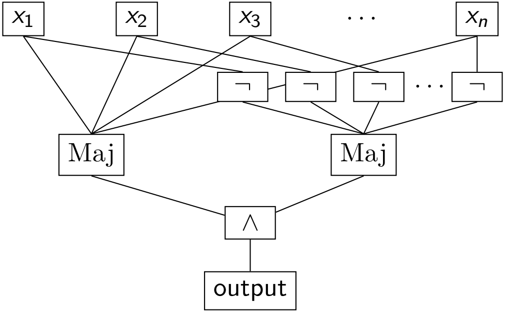

Word problem:

- Baumslag-Solitar groups $\mathbf{BS}_{1,q}$ (Robinson, 1993)
- nilpotent groups (Robinson, 1993)
- solvable linear groups (König, Lohrey, 2015)
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Group Theoretic Problems in TC^0

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Conjugacy problem:

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- nilpotent groups (Miasnikov, W., 2017)
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Because:

$w \in_G \{v\}^* \iff \mathrm{PP}(v, w) \geq 0$

Wreath products

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Multiplication: $$(b, f)(c, g) = (bc, f^c g)$$

for $b, c \in B$ and $f, g \in A^{(B)}$.

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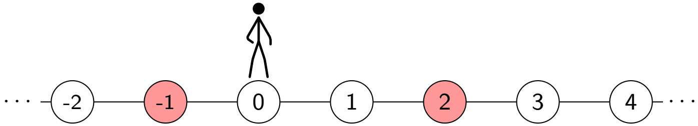

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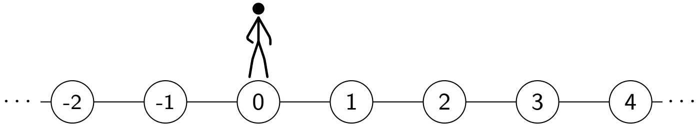

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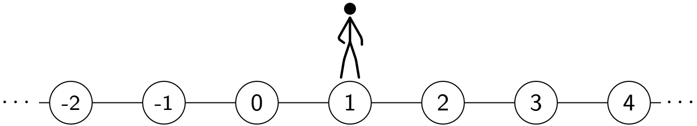

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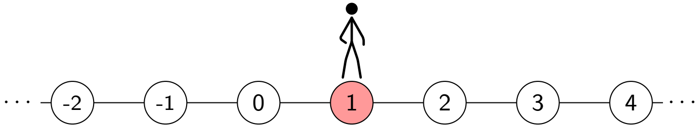

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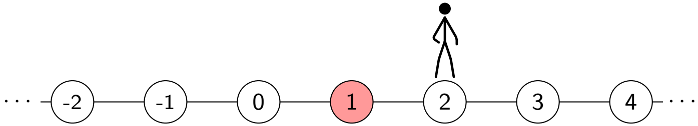

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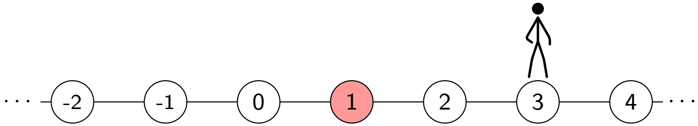

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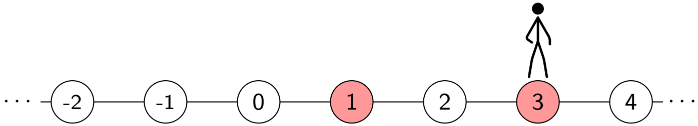

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$$\text{Elements of the form } (\underbrace{x}_{\text{position of lamplighter}}, \overbrace{\{y_1, \ldots, y_n\}}^{\text{lighted positions}})$$

- l, r generators of $\mathbb{Z}$ (with $l = r^{-1}$)
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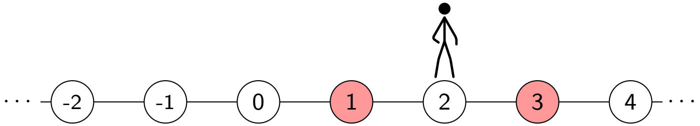

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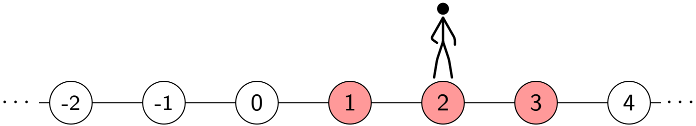

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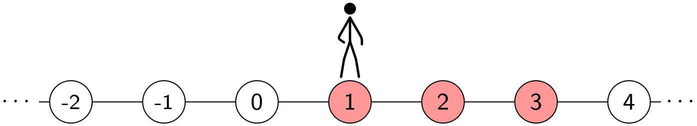

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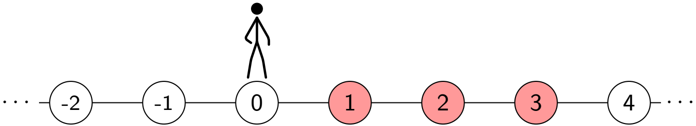

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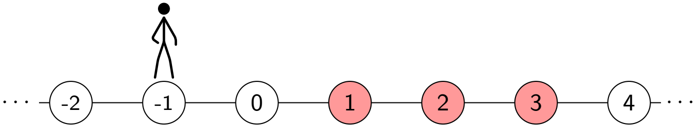

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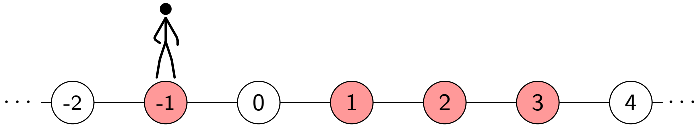

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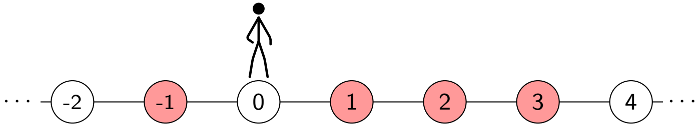

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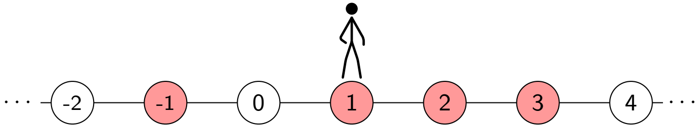

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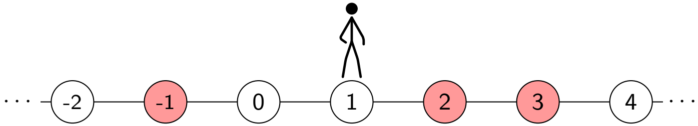

Word problem in wreath products

Lemma

Let A and B be f. g. groups. The following is in $\mathrm{AC}^0(\mathrm{WP}(A), \mathrm{WP}(B))$: On input $w \in \Sigma^*$, compute $(b, f) \in B \ltimes A^{(B)}$ with

$$w =_G (b, f).$$

Here f is given as $((b_1, a_1), \ldots, (b_n, a_n))$ with $\mathrm{supp}(f) = \{ b_1, \ldots, b_n \}$ and $a_i = f(b_i)$.

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Theorem

$\mathrm{WP}(A \wr B) \in \mathrm{AC}^0(\mathrm{WP}(A), \mathrm{WP}(B))$.

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Proof.

Denote π_B = projection onto B and let $w = w_1 \cdots w_n \in \Sigma^*$ be the input.

- Compute $b = \pi_B(w)$ and $\pi_B(w_{i+1} \cdots w_n)$ for all $i = 1, \ldots, n$

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- Define an equivalence relation $\approx$ on $\{1, \ldots, n\}$:

$$i \approx j \iff \pi_B(w_{i+1} \cdots w_n) =_B \pi_B(w_{j+1} \cdots w_n)$$

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- Let $[i]$ denote the equivalence class of i. Set

$$f_i = \begin{cases} \left(\pi_B(w_{i+1} \cdots w_n), \prod_{j \in [i]} w_j\right) & \text{if } i = \min[i], \\ (\varepsilon, \varepsilon) & \text{otherwise.} \end{cases}$$

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- Replace pairs $f_i = (b_i, a_i)$ by $(\varepsilon, \varepsilon)$ whenever $a_i =_A 1$
  $\rightsquigarrow$ use n oracle gates for $\mathrm{WP}(A)$ in parallel.

□

Word problem in iterated wreath products

Definition

Let A and B be groups and $d \in \mathbb{N}$,

left-iterated wreath product:

- $A\,{}^{1}\!\wr B = A \wr B$
- $A\,{}^{d}\!\wr B = A \wr (A\,{}^{d-1}\!\wr B)$

right-iterated wreath product:

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Corollary

- *Let A and B be f. g. abelian groups and let $d \in \mathbb{N}$. Then $\mathrm{WP}(A \wr^{d} B)$ and $\mathrm{WP}(A \wr^{d} B) \in \mathsf{TC}^0$.*
- *The word problem of a f. g. free solvable group is in TC^0.*

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A negative answer would imply that $\mathsf{TC}^0 \neq \mathsf{NC}^1$.

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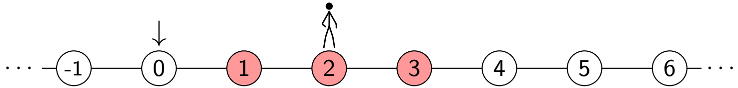

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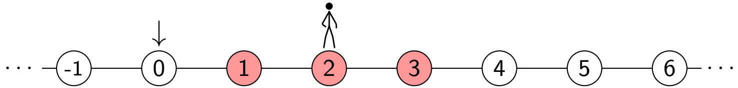

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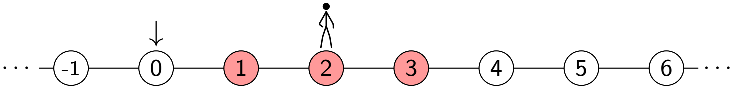

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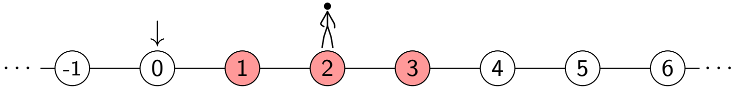

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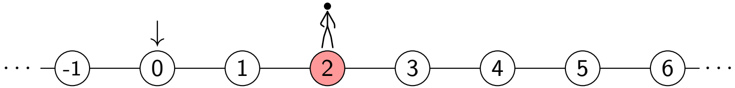

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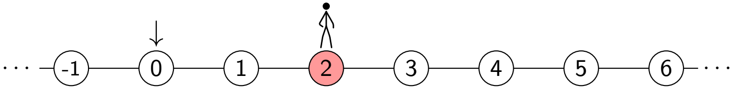

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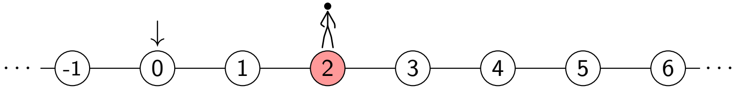

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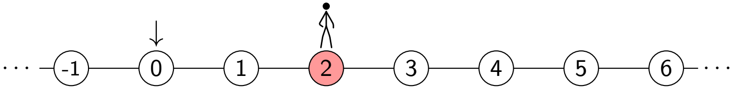

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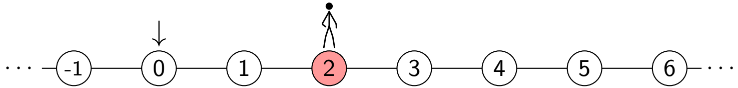

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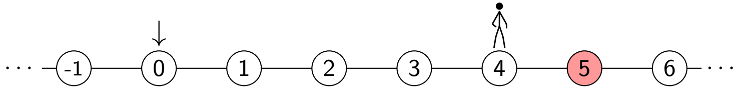

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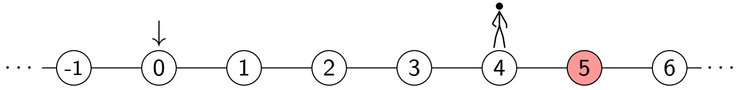

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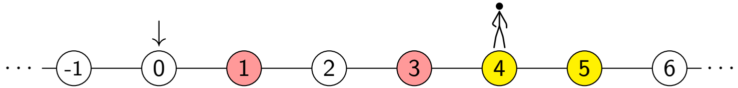

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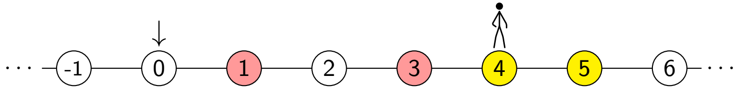

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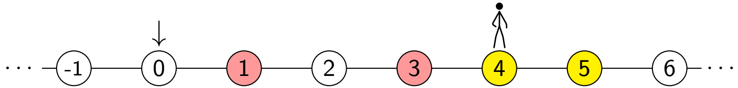

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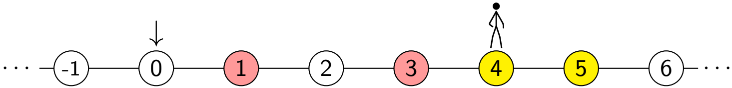

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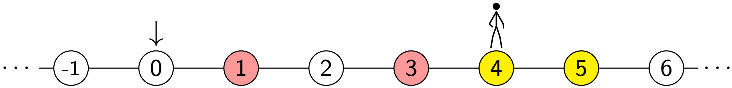

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Let A and B be arbitrary f. g. groups. Then,

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Conjugacy in wreath products

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Theorem

Let $\beta \in \mathbb{N}$ and suppose the orders of torsion elements in A are β-smooth. Then,

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Conjugacy in iterated wreath products

Theorem

Let $\beta \in \mathbb{N}$ and suppose the orders of torsion elements in A are β-smooth. Then,

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Corollary

- *Let A and B be f. g. abelian groups and let $d \in \mathbb{N}$. Then $\mathrm{CP}(A \wr^d B)$ and $\mathrm{CP}(A \wr^d B) \in \mathsf{TC}^0$.*
- *The conjugacy problem of a f. g. free solvable group is in TC^0.*

Open Questions

- Is $\mathrm{CP}(A \wr B) \in \mathsf{TC}^0(\mathrm{CP}(A), \mathrm{CP}(B), \mathrm{CSMMP}(B))$ in general?

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Thank you!