QuickXsort: Efficient Sorting with $n \log n - 1.399n + o(n)$ Comparisons on Average

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Sorting

Task: Sort a sequence of elements of some totally ordered universe using only pairwise comparisons.

$$\rightsquigarrow$$

QuickXsort

Hybrid algorithm: combination of Quicksort with some other algorithm X.

- QuickHeapsort (Cantone, Cincotti, CIAC 2000)
  (improvements: Diekert, W., CSR 2013)
- QuickWeakHeapsort (Edelkamp, Stiegeler, WAE 2000)
  (improvements: see proceedings)
- QuickMergesort (this talk)

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Objectives:

- Unified analysis of the average number of comparisons.

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Objectives:

- Unified analysis of the average number of comparisons.
- Pushing the limits towards optimal average case in place sorting ($n \log n + \kappa n + o(n)$ comparisons with κ as small as possible).

Constant-factor-optimal sorting with $n \log n + \kappa n + o(n)$ comparisons.

	Mem.	κ Worst	κ Avg.	κ Exper.
Information theo. lower bound		-1.44	-1.44	
QuickHeapsort	$\mathcal{O}(1)$	$\omega(1)$	-0.03	≈ 0.20
	$\mathcal{O}(n)$ bits	$\omega(1)$	-0.99	≈ -1.24
BottomUpHeapsort	$\mathcal{O}(1)$	$\omega(1)$	–	[0.35,0.39]
WeakHeapsort	$\mathcal{O}(n)$ bits	0.09	–	[-0.46,-0.42]
RelaxedWeakHeapsort	$\mathcal{O}(n)$	-0.91	-0.91	-0.91
ExternalWeakHeapsort #	$\mathcal{O}(n)$	-0.91	-1.26*	–
Mergesort	$\mathcal{O}(n)$	-0.91	-1.26	–
InPlaceMergesort	$\mathcal{O}(1)$	-1.25	–	–
Insertionsort	$\mathcal{O}(1)$	-0.91 †	-1.38 #	[-1.38,-1.39]
MergeInsertion	$\mathcal{O}(1)$	-1.32 †	-1.3999 #	[-1.43,-1.41]
QuickWeakHeapsort	$\mathcal{O}(n)$ bits	$\mathcal{O}(1)$ #	-0.91#	[-1.13,-1.25]
QuickMergesort #	$\mathcal{O}(1)$	$\mathcal{O}(1)$	-1.26	[-1.29,-1.27]
QuickMergesort (IS) #	$\mathcal{O}(1)$	$\mathcal{O}(1)$	-1.38	–
QuickMergesort (MI) #	$\mathcal{O}(1)$	$\mathcal{O}(1)$	-1.3999	[-1.41,-1.40]

in this paper, – not analyzed,

* only for $n = 2^k$, † needs $\Theta(n^2)$ moves.

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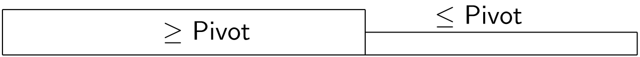
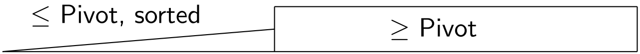
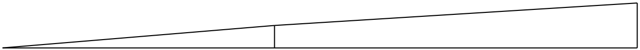


2. Sort one part with WeakHeapsort



3. Sort the other part recursively with QuickWeakHeapsort



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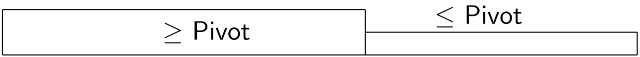

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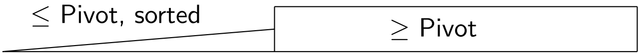

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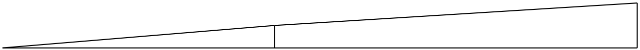

Advantage: WeakHeapsort can be implemented with less comparisons.

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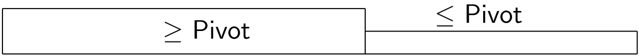

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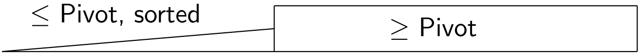

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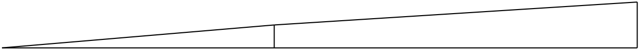

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Other point of view: in-place implementation of ExternalWeakHeapsort

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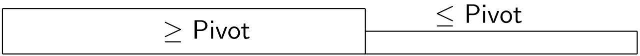

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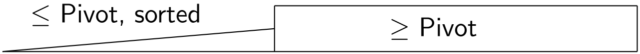

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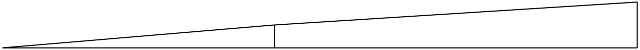

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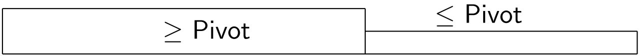

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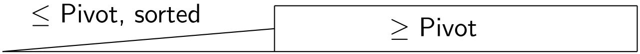

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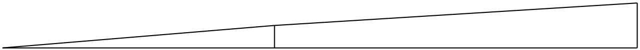

Advantage: during the sift down procedure only one comparison per level in the heap is needed.

$\rightsquigarrow$ $n \log n + \mathcal{O}(n)$ comparisons on average

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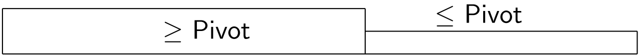

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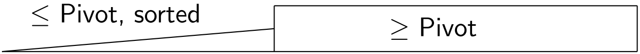

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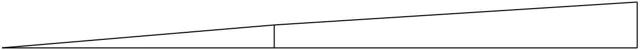

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Other point of view: in-place implementation of ExternalHeapsort

QuickMergesort

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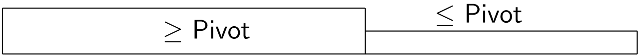

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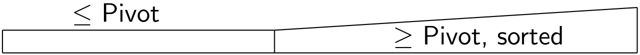

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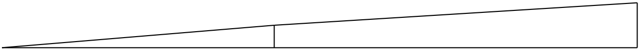

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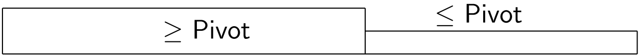

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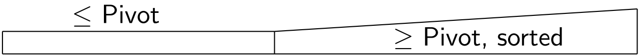

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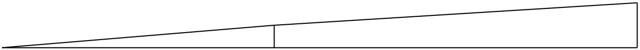
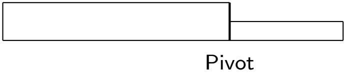


Advantage: Can be implemented in place.

Step 2: Mergesort in place

After partitioning



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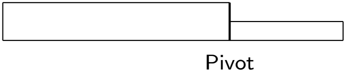

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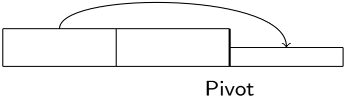

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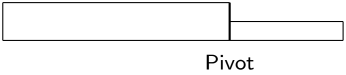

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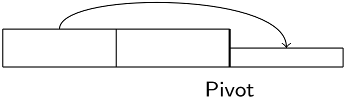

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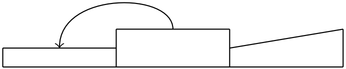

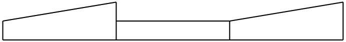

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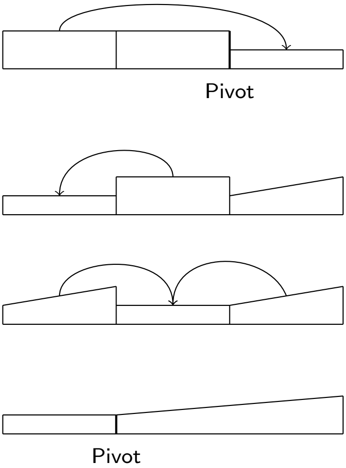

QuickMergesort

1. Partition the array according to some pivot element.
2. Sort one part with Mergesort.
3. Sort the remaining elements recursively with QuickMergesort.

5	1	8	7	6	2	3	10	9	11	12	4

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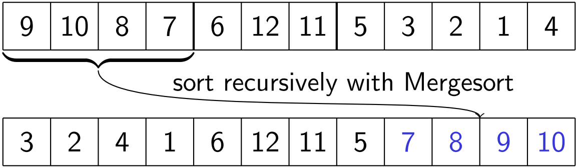

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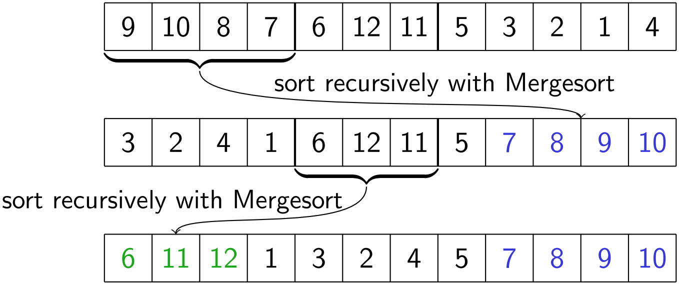

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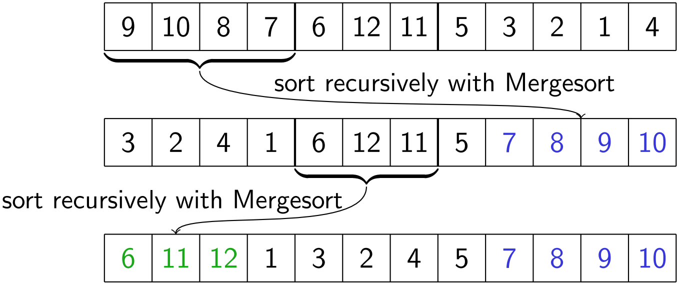

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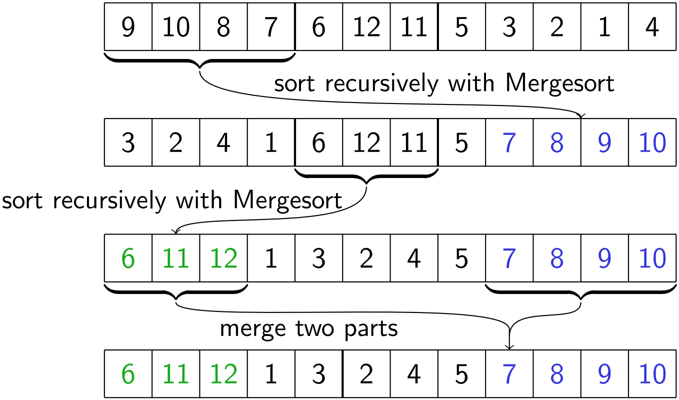

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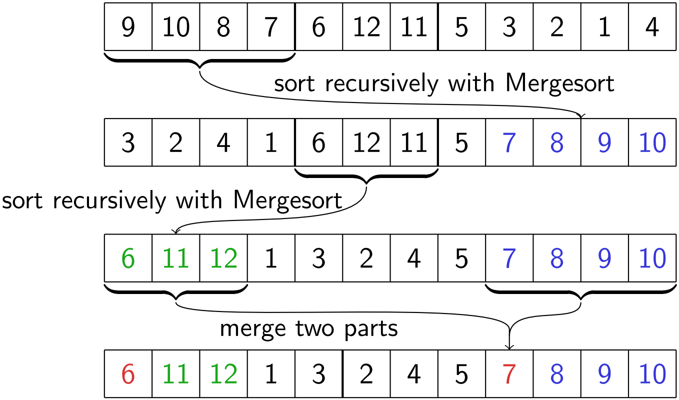

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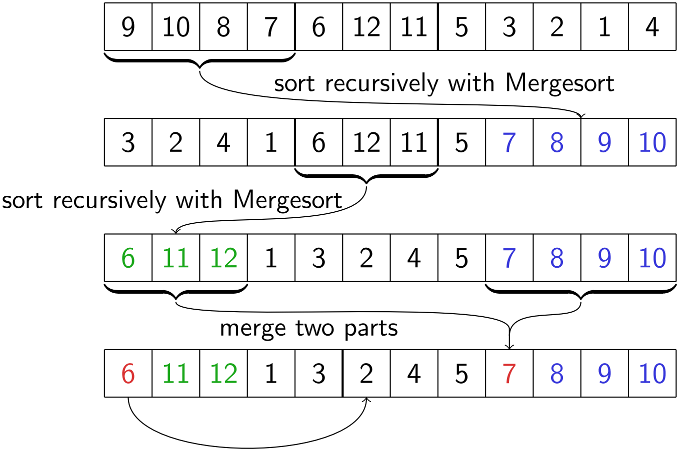

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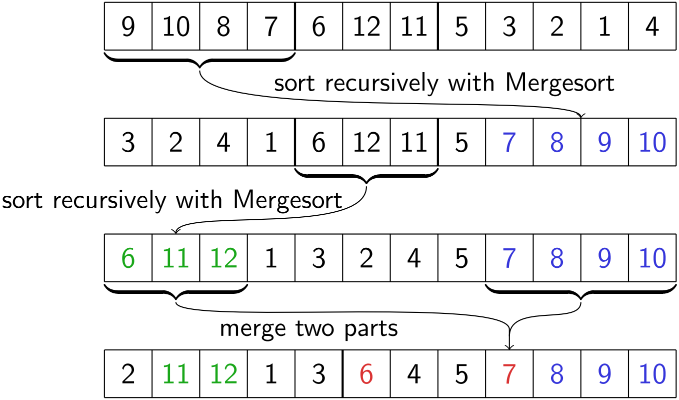

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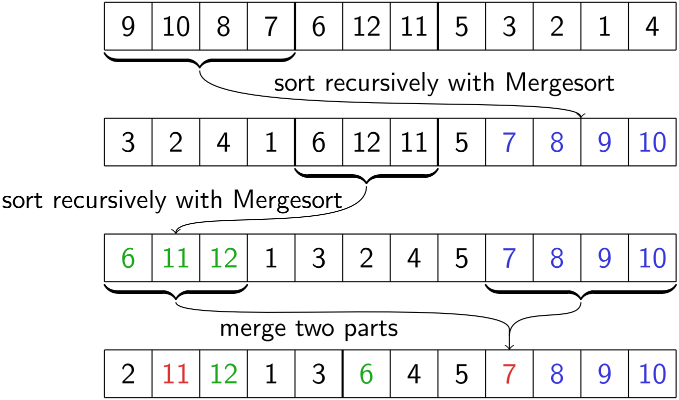

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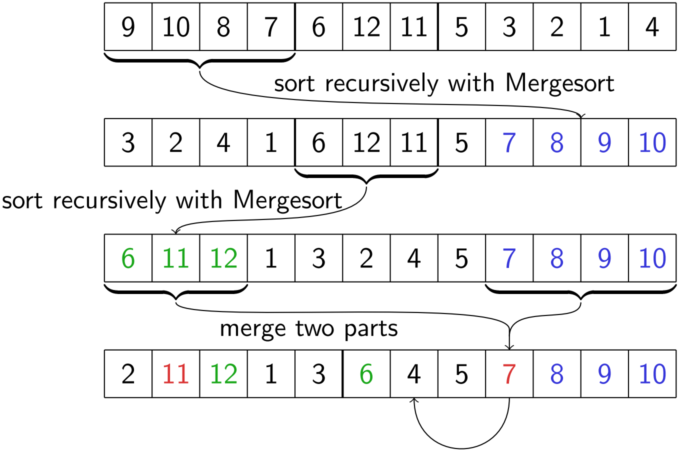

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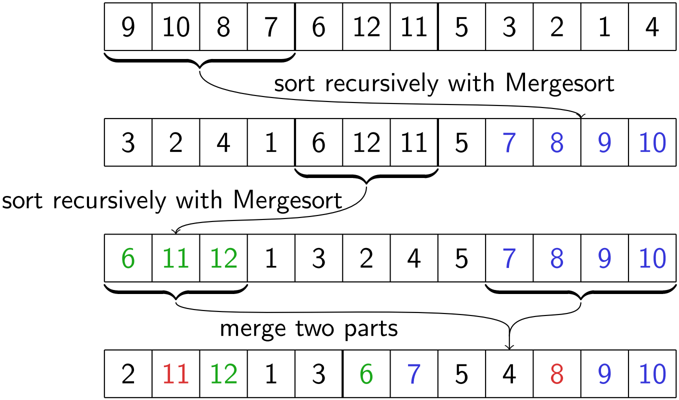

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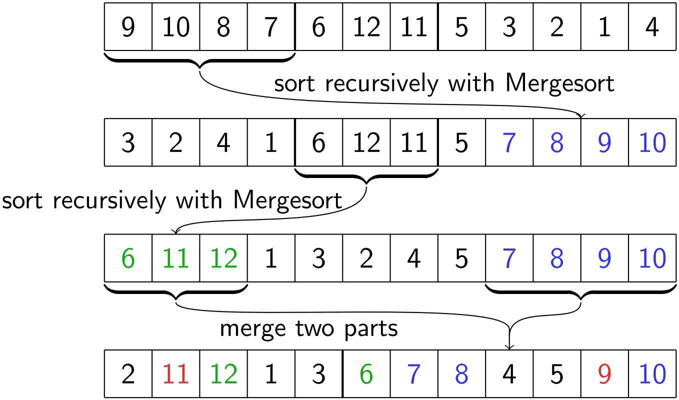

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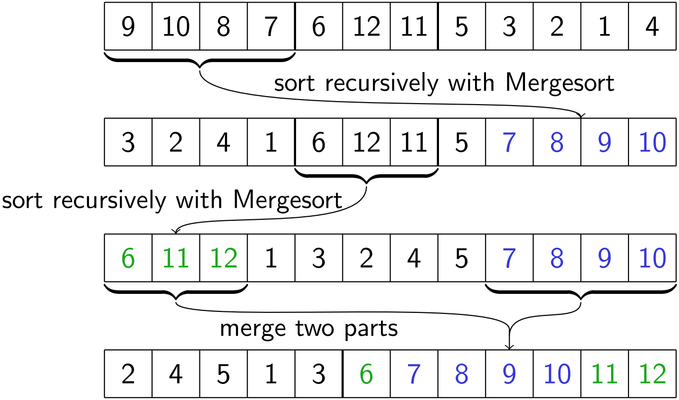

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9	10	8	7	6	12	11	5	3	2	1	4

sort recursively with Mergesort

3	2	4	1	6	12	11	5	7	8	9	10

sort recursively with Mergesort

6	11	12	1	3	2	4	5	7	8	9	10

merge two parts

2	4	5	1	3	6	7	8	9	10	11	12

sort recursively with QuickMergesort

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Theorem (QuickXsort Average-Case)

If X is some sorting algorithm requiring at most $n \log n + cn + o(n)$ comparisons on average, then QuickXsort needs at most $n \log n + cn + o(n)$ comparisons on average.

Proof idea

General recurrence relation for the average number of comparisons: $(S(n) = n \log n + cn + o(n)$ = bound for the average number of comparisons of X)

$$T(n) \le T_{\mathrm{pivot}}(n) + n + \sum_{k=1}^{n} \Big(\Pr[\text{pivot } = k] \cdot \max\{\, T(k-1) + S(n-k), T(n-k) + S(k-1) \,\} \Big)$$

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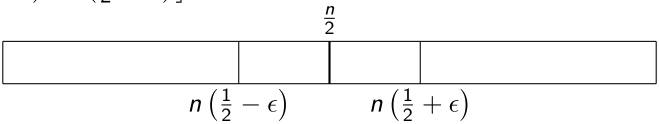
It is very unlikely that the pivot is chosen outside the interval $\left[n\left(\frac{1}{2} - \epsilon\right), n\left(\frac{1}{2} + \epsilon\right)\right]$.



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&\approx T_{\text{pivot}}(n) + n + T(n/2) + S(n/2) \\
&= n + n\log(n/2) + cn + o(n).
\end{aligned}$$

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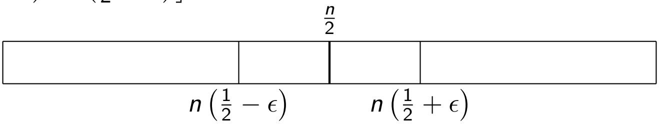

Worst case of QuickXsort

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Trick to obtain a provable bound for the worst case (similar to Introsort (Musser, 1997)):

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Theorem (QuickXsort worst case)

Let X be a sorting algorithm with at most $n \log n + cn + o(n)$ comparisons on average and $n \log n + \mathcal{O}(n)$ comparisons in the worst case. Then QuickXsort *(with the above modification) performs at most $n \log n + cn + o(n)$ comparisons on average and $n \log n + \mathcal{O}(n)$ comparisons in the worst case.*

Corollary

QuickWeakHeapsort *performs at most* $n \log n - 0.91n + o(n)$ *comparisons on average.*

Corollary

QuickMergesort *is an internal sorting algorithm that performs at most* $n \log n - 1.26n + o(n)$ *comparisons on average.*

(See e.g. Knuth, *The Art of Computer Programming, Sorting and Searching* 5.2.4–13.)

QuickMergesort with base case

Further improvement for QuickMergesort:

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Theorem

Let Z be some sorting algorithm with $n \log n + dn + o(n)$ *comparisons on average and at most* $\mathcal{O}(n^2)$ *other operations (e.g. moves). If base cases of size* $\Theta(\log n)$ *are sorted with Z,* QuickMergesort *needs at most* $n \log n + dn + o(n)$ *comparisons on average and* $\mathcal{O}(n \log n)$ *other instructions.*

QuickMergesort with base case InsertionSort

Insert the elements successively into the already sorted sequence:

- find the position of each element by binary search
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InsertionSort *needs* $n \log n - (2 \ln 2 + c(n)) \cdot n + \mathcal{O}(\log n)$ *comparisons on average where* $c(n) \in [-0.005, 0.005]$.

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Corollary

QuickMergesort *with base case* InsertionSort *uses at most* $n \log n - 1.38n + o(n)$ *comparisons on average.*

The algorithm:

1. Build pairs $a_i > b_i$.
2. Sort the values $a_1,\ldots,a_{\lfloor n/2 \rfloor}$ recursively.
3. Insert the elements $b_1 \ldots, b_{\lceil n/2 \rceil}$ into the linear chain by binary insertion following a special ordering.

Theorem (Hadian 1969, Knuth 1973)

MergeInsertion *needs at most* $n \log n - 1.329n + \mathcal{O}(\log n)$ *comparisons in the worst case.*

QuickMergesort with base case MergeInsertion

Theorem (Average Case of MergeInsertion)

A simplified version of MergeInsertion *needs at most* $n \log n - 1.3999 \cdot n + \mathcal{O}(\log n)$ *comparisons on average.*

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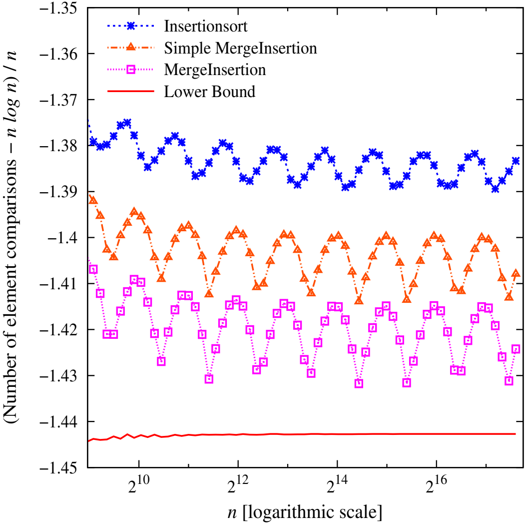
Corollary

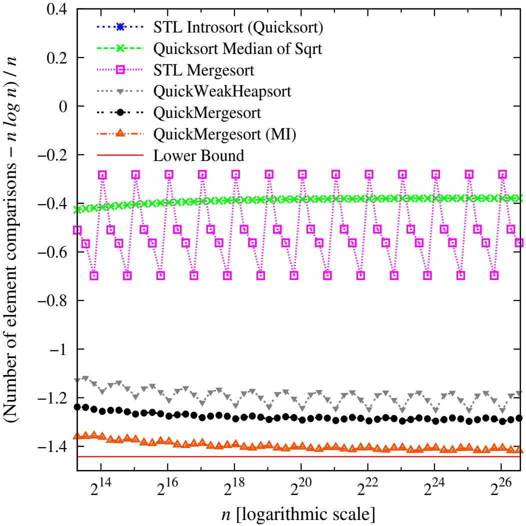
QuickMergesort *with* MergeInsertion *as base case needs at most* $n \log n - 1.3999n + o(n)$ *comparisons on average.*

Experiments on InsertionSort and MergeInsertion



Sorting with $n \log n + \kappa n$ comparisons.

Experimental behavior of the linear term of QuickXsort



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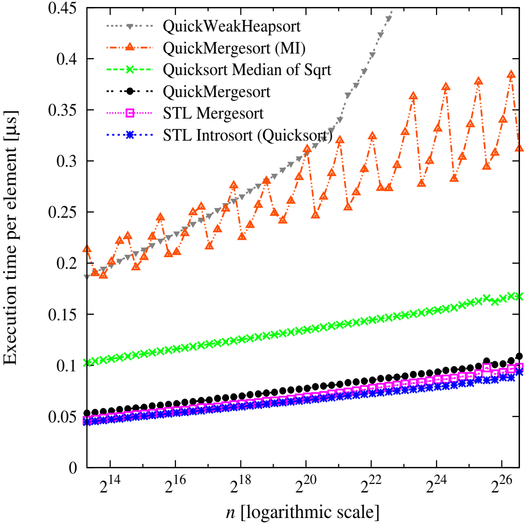
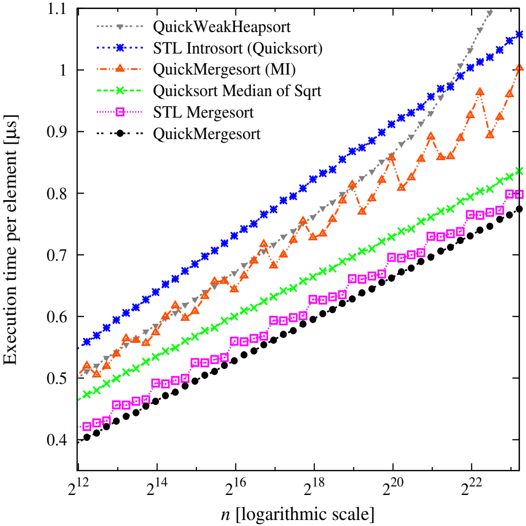
Running times of QuickXsort and other algorithms



Running time with integer data.

Running times of QuickXsort and other algorithms



Running time with more expensive comparisons simulated by calculating the logarithm of one operand in every comparison.

Constant-factor-optimal sorting with $n \log n + \kappa n + o(n)$ comparisons.

	Mem.	κ Worst	κ Avg.	κ Exper.
Information theo. lower bound		-1.44	-1.44	
BottomUpHeapsort	$\mathcal{O}(1)$	$\omega(1)$	–	[0.35,0.39]
QuickHeapsort	$\mathcal{O}(1)$	$\omega(1)$	-0.03	$\approx$ 0.20
	$\mathcal{O}(n)$ bits	$\omega(1)$	-0.99	$\approx$ -1.24
WeakHeapsort	$\mathcal{O}(n)$ bits	0.09	–	[-0.46,-0.42]
RelaxedWeakHeapsort	$\mathcal{O}(n)$	-0.91	-0.91	-0.91
ExternalWeakHeapsort #	$\mathcal{O}(n)$	-0.91	-1.26*	–
Mergesort	$\mathcal{O}(n)$	-0.91	-1.26	–
InPlaceMergesort	$\mathcal{O}(1)$	-1.25	–	–
InsertionSort	$\mathcal{O}(1)$	-0.91 †	-1.38 #	[-1.38,-1.39]
MergeInsertion	$\mathcal{O}(1)$	-1.32 †	-1.3999 #	[-1.43,-1.41]
QuickWeakHeapsort	$\mathcal{O}(n)$ bits	$\mathcal{O}(1)$ #	-0.91#	[-1.13,-1.25]
QuickMergesort #	$\mathcal{O}(1)$	$\mathcal{O}(1)$	-1.26	[-1.29,-1.27]
QuickMergesort (IS) #	$\mathcal{O}(1)$	$\mathcal{O}(1)$	-1.38	–
QuickMergesort (MI) #	$\mathcal{O}(1)$	$\mathcal{O}(1)$	-1.3999	[-1.41,-1.40]

in this paper, – not analyzed,

* only for $n = 2^k$, † needs $\Theta(n^2)$ moves.

Conclusion and open problems

- Generic analysis of QuickXsort.
- QuickMergesort is highly competitive.
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Thank you!