Conjugacy in Baumslag-Solitar groups

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Bordeaux, June 15, 2015

Dehn's fundamental problems

Let G be a group, generated by a finite set Σ with $\Sigma = \Sigma^{-1} \subseteq G$. Write $\overline{a}$ for $a^{-1} \in \Sigma$.

- Word problem: Given $w \in \Sigma^*$. Question: Is $w = 1$ in G?
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Structure of the talk:

- Overview
- Word problem
- Conjugacy problem
- Generalized Baumslag-Solitar groups

Overview: Classification of Baumslag-Solitar groups

Baumslag-Solitar group:

$$\mathbf{BS}_{p,q} = \left\langle a, t \mid ta^p t^{-1} = a^q \right\rangle = \mathrm{HNN}(\langle a \rangle, t; a^p \mapsto a^q)$$

W.l.o.g. $1 \le p \le |q|$.

Baumslag-Solitar group:

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- G is solvable $\iff p = 1$,
- G is linear $\iff p = |q|$ or $p = 1$,
- G is not linear, otherwise.

Overview: Word and conjugacy problem

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Theorem (Robinson, 1993)

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TC^0 = recognized by a family of circuits of constant depth with unbounded fan-in $\neg$, $\wedge$, $\vee$, and majority gates.

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Theorem (Diekert, Miasnikov, W., 2014)

The word and conjugacy problem of $\mathbf{BS}_{1,q}$ *are (uniform)* TC^0*-complete.*

Theorem (Lipton, Zalcstein, 1977 / Simon, 1979)

The word problem of linear groups (in particular for linear Baumslag-Solitar groups) can be solved in LOGSPACE.

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The word and conjugacy problem of $\mathbf{BS}_{p,q}$ *is in* LOGDCFL.

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Conjecture (W., 2014)

The conjugacy problem of $\mathbf{BS}_{p,q}$ is in LOGSPACE.

Proof: the word problem of $\mathbf{BS}_{1,2}$ is in TC^0

$$\mathbf{BS}_{1,2} \cong \mathbb{Z}[1/2] \rtimes \mathbb{Z} = \{\, (r, m) \mid r \in \mathbb{Z}[1/2], m \in \mathbb{Z} \,\}$$

with multiplication

$$(r, m) \cdot (s, q) = (r + 2^m s, m + q).$$

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The isomorphism is given by

$$a \mapsto (1, 0), \qquad t \mapsto (0, 1).$$

Example

$$tataa\bar{t} \mapsto (0,1)(1,0)(0,1)(1,0)(1,0)(0,-1)$$

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Let $(r_1, m_1), \ldots, (r_n, m_n) \in \mathbb{Z}[1/2] \rtimes \mathbb{Z}$. Then, for $(r, m) = (r_1, m_1) \cdots (r_n, m_n)$, we have $m = \sum_{i=1}^{n} m_i$ and

$$r = \sum_{i=1}^{n} r_i \cdot \prod_{j=1}^{i-1} 2^{m_j}$$

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Corollary (Diekert, Miasnikov, W., 2014)

The word problem of $\mathbf{BS}_{1,q}$ is in uniform TC^0.

Proof: iterated addition and iterated multiplication (Hesse, 2001) is in uniform TC^0.

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Theorem (König, Lohrey, 2015)

The word problem of f.g. solvable linear groups is in uniform TC^0.

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$\mathbf{BS}_{p,q}$ contains a free subgroup $\left\langle t, ata^{-1} \right\rangle$ if $|p|, |q| \neq 1$.

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Two aspects:

- Word problem of solvable Baumslag-Solitar groups.
- Word problem of the free group F_2.

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Britton's Lemma

$w \in \langle a \rangle = A$ in $\mathbf{BS}_{p,q} \iff w$ can be reduced to some word in $\{a, \overline{a}\}^*$ by Britton reductions

$$t^{\varepsilon} a^{k} t^{-\varepsilon} \to a^{\ell} \qquad (\varepsilon \in \{\pm 1\}).$$

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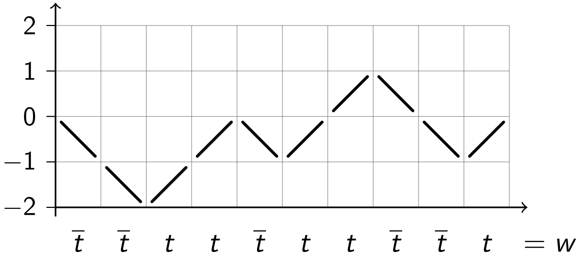
$\rightsquigarrow$ word problem in P by storing exponents in binary.

The word problem of $\mathbf{BS}_{p,q}$

Consider the subgroup $\langle t \rangle$ (= quotient $a \mapsto 1$, $t \mapsto t$):



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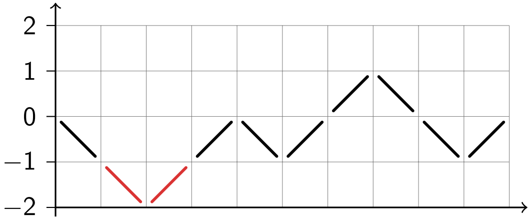

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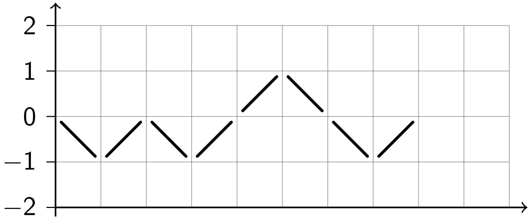

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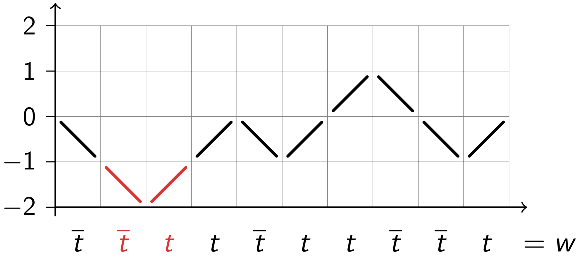

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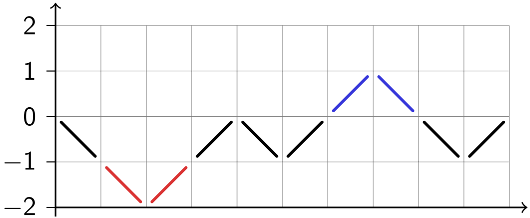

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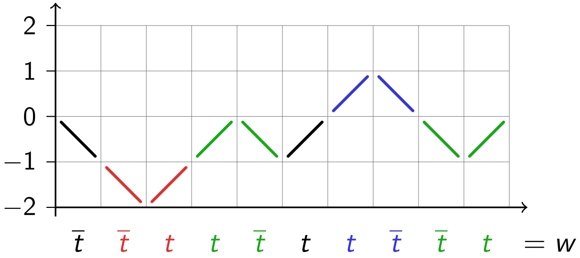

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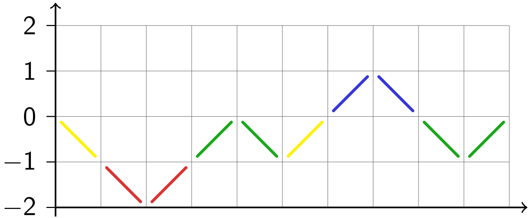

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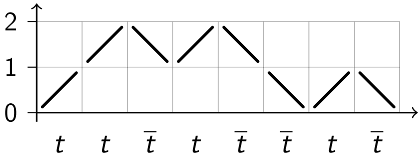

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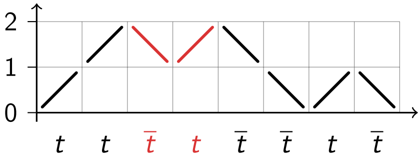

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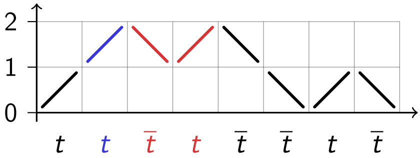

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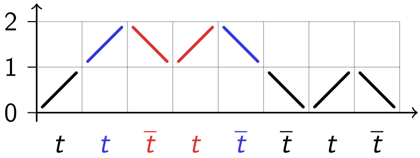

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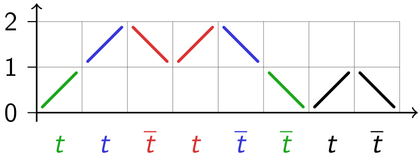

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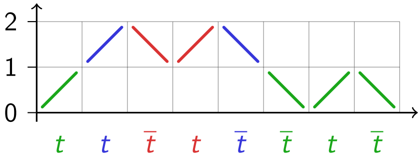

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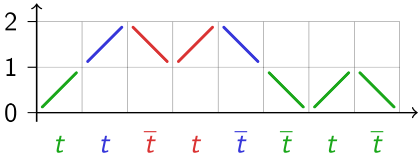

$\rightsquigarrow w \in \langle a \rangle = A$

The word problem of $\mathbf{BS}_{p,q}$

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$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_i} \underbrace{a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}} t^{\varepsilon_{j+1}} a^{k_{j+1}} \cdots t^{\varepsilon_n} a^{k_n}$$

with $\varepsilon_\mu \in \{ \pm 1 \}$, $k_\mu \in \mathbb{Z}$. Define

$$w_{i,j} = a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}$$

$$k_{i,j} = \sum_{\nu=i}^{j} k_\nu \cdot \prod_{\mu=i+1}^{\nu} \left(\frac{q}{p}\right)^{\varepsilon_\mu} \quad \in \mathbb{Z}[1/pq]$$

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Lemma 1

$$w_{i,j} \in A \iff w_{i,j} = a^{k_{i,j}} \text{ in } \mathbf{BS}_{p,q}$$

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$$w_{i,j} = a^{k_i} t^{\varepsilon_{i+1}} a^{k_{i+1}} \cdots t^{\varepsilon_j} a^{k_j}$$

$$k_{i,j} = \sum_{\nu=i}^{j} k_\nu \cdot \prod_{\mu=i+1}^{\nu} \left(\frac{q}{p}\right)^{\varepsilon_\mu} \qquad \in \mathbb{Z}[1/pq]$$

Numbers $k_{i,j}$ can be computed in TC^0.

Lemma 1

$$w_{i,j} \in A \iff w_{i,j} = a^{k_{i,j}} \text{ in } \mathbf{BS}_{p,q}$$

Proof.

Induction: by Britton's Lemma, $w = a^{k_0} t^{\varepsilon_1} w' t^{-\varepsilon_1} w''$. □

The word problem of $\mathbf{BS}_{p,q}$

Define a relation $\sim_{\mathcal{C}} \subseteq \{1,\ldots,n\} \times \{1,\ldots,n\}$:
For $i < j$:

$$i \sim_{\mathcal{C}} j \iff \varepsilon_i = -\varepsilon_j \text{ and } \sum_{\ell=i+1}^{j-1} \varepsilon_\ell = 0 \quad \text{(same level)}$$

$$\text{and } k_{i,j-1} \in \begin{cases} p\mathbb{Z} & \text{if } \varepsilon_i = 1 \\ q\mathbb{Z} & \text{if } \varepsilon_i = -1. \end{cases}$$

For $i > j$: $i \sim_{\mathcal{C}} j \iff j \sim_{\mathcal{C}} i$.

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Lemma 2

If $i \approx j$ and $\varepsilon_i = -\varepsilon_j$, then $i \sim_{\mathcal{C}} j$.

Proof.

Show: $i \sim_{\mathcal{C}} \ell$, $\ell \sim_{\mathcal{C}} m$, and $m \sim_{\mathcal{C}} j \implies i \sim_{\mathcal{C}} j$. Then induction.

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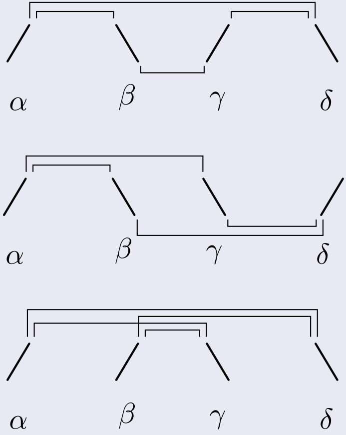
Let $\{\alpha, \beta, \gamma, \delta\} = \{i, j, \ell, m\}$ with $\alpha < \beta < \gamma < \delta$.



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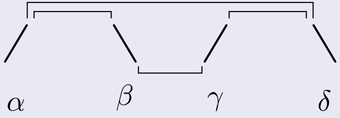

$$k_{\alpha,\delta-1} = k_{\alpha,\beta-1} + \frac{p}{q} \cdot k_{\beta,\gamma-1} + k_{\gamma,\delta-1}$$

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Let $\Sigma_w = \{ t_{[i]}, \overline{t}_{[i]} \mid i \in \{1, \dots, n\} \}$ be a new set of generators:

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Corollary

$$w = 1 \text{ in } \mathbf{BS}_{p,q} \iff \widetilde{w} = 1 \text{ in } F(\Sigma_w) \text{ and } k_{0,n} = 0.$$

Proof of Lemma 3.

Let $w_{i,j} \in \langle a \rangle = A$. By Britton's Lemma,

$$w_{i,j} = a^{k_i}\, t^{\varepsilon_{i+1}} w_{i+1,\ell-1} t^{\varepsilon_\ell}\, w_{\ell,j}$$

with $\varepsilon_\ell = -\varepsilon_{i+1}$, $w_{\ell,j} \in A$, and

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Thus, $\widetilde{w}_{i,j} = t_{[i+1]}^{\varepsilon_{i+1}}\, \widetilde{w}_{i+1,\ell-1}\, t_{[i+1]}^{-\varepsilon_{i+1}}\, \widetilde{w}_{\ell,j} = 1$ in $F(\Sigma_w)$.

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Theorem (W., today)

The word problem of Baumslag-Solitar groups is in LOGSPACE.

Solving the conjugacy problem of $\mathbf{BS}_{p,q}$

Input: $v, w \in \{a, \overline{a}, t, \overline{t}\}^*$.

1. Compute Britton-reduced words $\hat{v}, \hat{w}$.
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Britton reductions in LOGSPACE:

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Example

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2. Compute cyclically Britton-reduced words $\tilde{v}, \tilde{w}$.
3. Check whether $\tilde{v} \sim \tilde{w}$ using Collins' Lemma.

Britton reductions in LOGSPACE:

$$w = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_n} a^{k_n} \in \mathbf{BS}_{p,q},$$

For $i = 0, \ldots, n$

- Find the largest $j > i$ with $w_{i,j-1} = a^{k_{i,j-1}}$ in $\mathbf{BS}_{p,q}$,
- Output $a^{k_{i,j-1}} t^{\varepsilon_j}$,
- $i := j$.

Example

$$w = \overline{a}\, t\, a\, t\, aa\, \overline{t}\, aa\, \overline{t}\, t\, a\, \overline{t}\, t\, a\, t\, a \in \mathbf{BS}_{2,3}$$

$\uparrow \quad \uparrow$

Output $= a^8 t\, aat$

Solving the conjugacy problem of $\mathbf{BS}_{p,q}$

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$$\text{Output} = a^8 t\, aat\, a$$

Solving the conjugacy problem of $\mathbf{BS}_{p,q}$

Let $g = a^k \in \langle a \rangle$. Then

$$aga^{-1} = g,$$
$$tgt^{-1} = ta^k t^{-1} \begin{cases} = a^{\frac{q}{p}k} & \text{if } p \mid k \\ \text{is Britton reduced} & \text{otherwise.} \end{cases}$$

Thus, for $k \neq \ell$:

$$a^k \sim a^\ell \iff \exists j \in \mathbb{Z} \text{ such that } k \cdot \left(\frac{q}{p}\right)^j = \ell$$
$$\text{and } \begin{cases} k \in p\mathbb{Z},\ \ell \in q\mathbb{Z}, & \text{if } j > 0, \\ k \in q\mathbb{Z},\ \ell \in p\mathbb{Z}, & \text{otherwise.} \end{cases}$$

There are only polynomially many possibilities for j
$\rightsquigarrow$ check them all in parallel.

Corollary

It can be checked in TC^0 *whether* $a^k \sim a^\ell$.

Solving the conjugacy problem of $\mathbf{BS}_{p,q}$

Lemma (Collin's Lemma for HNN extensions)

Let $v, w \in \{a, \overline{a}, t, \overline{t}\}^*$ *be*

- *cyclically Britton-reduced,*
- $v, w \notin \langle a \rangle$.

Then

$$v \sim w \iff \text{there is a cyclic permutation } w' \text{ of } w \text{ and } x \in \mathbb{Z} \text{ such that } v = a^x w' a^{-x}.$$

Solving the conjugacy problem of $\mathbf{BS}_{p,q}$

- Test all cyclic permutations in parallel
- For
$$w' = a^{k_0} t^{\varepsilon_1} a^{k_1} \cdots t^{\varepsilon_n} a^{k_n} \in \mathbf{BS}_{p,q},$$
$$v = a^{\ell_0} t^{\varepsilon_1} a^{\ell_1} \cdots t^{\varepsilon_n} a^{\ell_n} \in \mathbf{BS}_{p,q},$$
the existence of $x \in \mathbb{Z}$ with $v = a^x w' a^{-x}$ reduces to finding an integral solution $x, y_1, \ldots, y_n$ for the system of equations
$$y_i = \frac{1}{\alpha_i}\left(x \cdot \prod_{\mu=1}^{i-1} \left(\frac{p}{q}\right)^{\varepsilon_\mu} + \sum_{\nu=1}^{i-1} (k_\nu - \ell_\nu) \cdot \prod_{\mu=\nu+1}^{i-1} \left(\frac{p}{q}\right)^{\varepsilon_\mu} \right),$$
$$x = k_n - \ell_n + x \cdot \prod_{\mu=1}^{n} \left(\frac{p}{q}\right)^{\varepsilon_\mu} + \sum_{\nu=1}^{n-1} (k_\nu - \ell_\nu) \cdot \prod_{\mu=\nu+1}^{n} \left(\frac{p}{q}\right)^{\varepsilon_\mu}.$$
- Can be done in TC^0.

Theorem (W.)

Let G be a Baumslag-Solitar group. Then the conjugacy problem of G is

- TC^0*-complete if* $G = \mathbf{BS}_{1,p}$ *is a solvable Baumslag-Solitar group,*
- *in* LOGSPACE*, otherwise.*

Generalized Baumslag-Solitar groups

A generalized Baumslag-Solitar group (GBS group) is a

- fundamental group of a finite graph of groups
- with infinite cyclic vertex and edge groups.

A GBS group G is given by a graph of groups $\mathcal{G}$:

- an undirected graph (V, E)
  (with involution $\overline{\cdot} : E \to E$, $\iota(t)$ the initial, $\tau(t)$ the terminal vertex of $t \in E$),
- $\alpha_t, \beta_t \in \mathbb{Z} \setminus \{ 0 \}$ for $t \in E$ such that $\alpha_t = \beta_{\overline{t}}$.

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$$F(\mathcal{G}) = \left\langle V, E \;\middle|\; \bar{t}t = 1, tb^{\beta_t}\bar{t} = a^{\alpha_t} \text{ for } t \in E, a = \iota(t), b = \tau(t) \right\rangle$$

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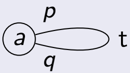
Fix a vertex $a \in V$: $G = \pi_1(\mathcal{G}, a) \leq F(\mathcal{G})$

$$G = \{ a_0 t_1 a_1 \cdots t_n a_n \mid t_i \in E, a_i = \tau(t_i) = \iota(t_{i+1}), a_0 = a_n = a \}$$
$= $ "all closed paths starting at a."

Generalized Baumslag-Solitar groups

Example

$\mathbf{BS}_{p,q}$



Example

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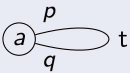

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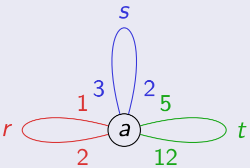

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\bar{r} = a^2, sa^2\bar{s} = a^3, ta^{12}\bar{t} = a^5 \right\rangle$$

Word and Conjugacy Problem in GBS groups

- Word problem
- Britton reductions
- Conjugacy for cyclically reduced words $u, v \notin \langle a \rangle$

work all as for ordinary Baumslag-Solitar groups.
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work all as for ordinary Baumslag-Solitar groups.
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- But: Conjugacy for cyclically reduced words $u, v \in \langle a \rangle$ does not work as for ordinary Baumslag-Solitar groups.

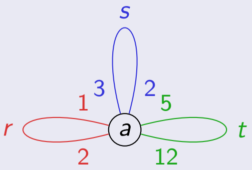
Remember:

$$a^k \sim a^\ell \text{ in } \mathbf{BS}_{p,q} \iff \exists j \in \mathbb{Z} \text{ with } k \cdot \left(\frac{q}{p}\right)^j = \ell \text{ and...}$$

Now: more than polynomially many potential conjugating elements.

Example

$$G = F(\mathcal{G}) = \left\langle a, r, s, t \;\middle|\; ra\overline{r} = a^2, sa^2\overline{s} = a^3, ta^{12}\overline{t} = a^5 \right\rangle$$



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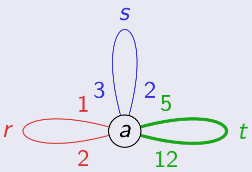

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$$\overline{t}\, a^{15}\, t \quad = \quad a^{36}$$

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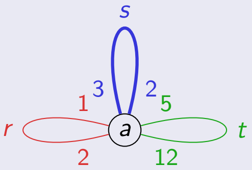

$$a^{15} \sim a^{16}?$$

$$\overline{s}\overline{t}\, a^{15}\, ts = \overline{s}\, a^{36}\, s = a^{24}$$

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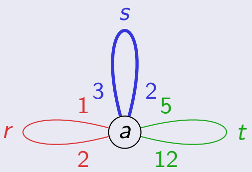

$$a^{15} \sim a^{16}?$$

$$\overline{s}\,\overline{s}\,\overline{t}\; a^{15}\; tss = \overline{s}\,\overline{s}\; a^{36}\; ss$$

$$= a^{16}$$

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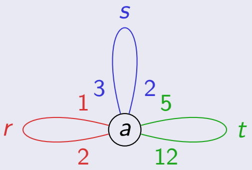

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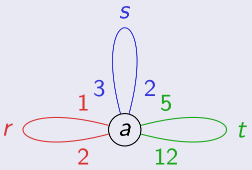

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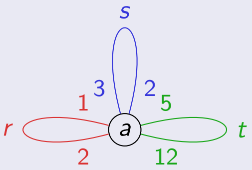

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Question: $a^k \sim a^\ell$? Write $k = r_k \cdot 2^c \cdot 3^d \cdot 5^e$,

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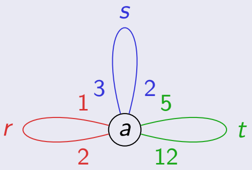

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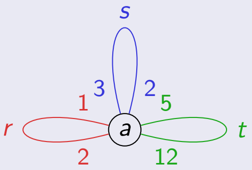

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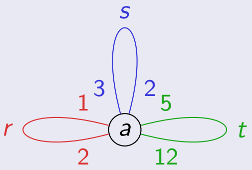

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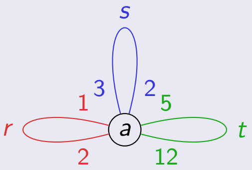

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$\rightsquigarrow$ suffices to consider (c, d, e).

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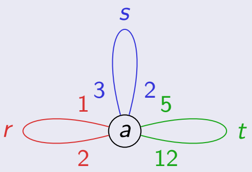

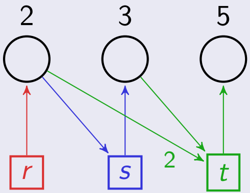

+ inverse transitions

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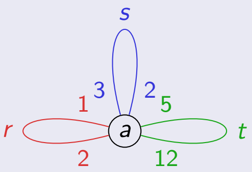

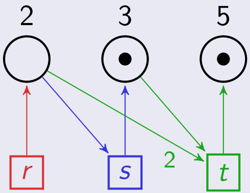

$+$ inverse transitions

a^{15} $\qquad (0, 1, 1)$

Example

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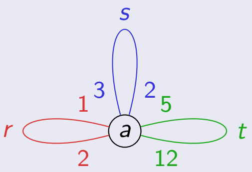

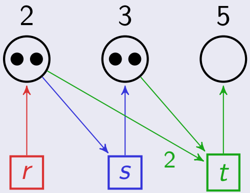

+ inverse transitions

a^{15}	$(0, 1, 1)$
$\bar{t}\, a^{15}\, t$	$(2, 2, 0)$

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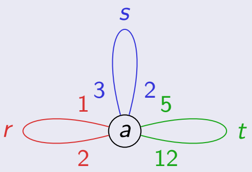

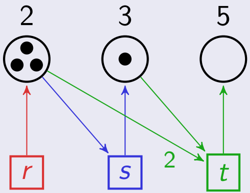

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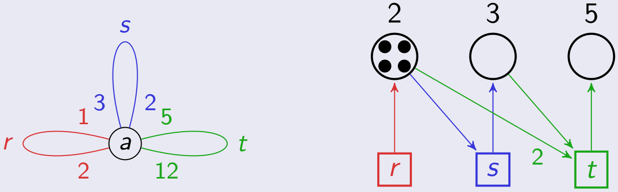

+ inverse transitions

a^{15}	$(0,1,1)$
$\overline{t}\,a^{15}\,t$	$(2,2,0)$
$\overline{s}\overline{t}\,a^{15}\,ts$	$(3,1,0)$
$a^{16} = \overline{s}\overline{s}\overline{t}\,a^{15}\,tss$	$(4,0,0)$

Question: $a^k \sim a^\ell$?
Let $\mathcal{P} = \{\text{primes occurring in } \alpha_t, \beta_t (t, \in E)\}$.

$$k = r_k \cdot \prod_{p \in \mathcal{P}} p^{e_p(k)}, \qquad \ell = r_\ell \cdot \prod_{p \in \mathcal{P}} p^{e_p(\ell)}.$$

If $r_k \neq r_\ell$, then $a^k \not\sim a^\ell$. Otherwise,

$$a^k \sim a^\ell \iff (e_p(k))_{p \in \mathcal{P}} \approx (e_p(\ell))_{p \in \mathcal{P}}$$

$\approx =$ congruence on $\mathbb{N}^{\mathcal{P}}$ generated by $(e_p(\alpha_t))_{p \in \mathcal{P}} \approx (e_p(\beta_t))_{p \in \mathcal{P}}$ for $t \in E$.

Word and Conjugacy Problem in GBS groups

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Theorem (Ballantyne, Lankford, 1981)

There is a weight-reducing, confluent rewriting system for $\approx$.

Writing down $(e_p(k))_{p \in \mathcal{P}}$ takes space $\mathcal{O}(\log \log k)$.
Greedy application of rewriting rules $\rightsquigarrow$ LOGSPACE.

Conjugacy in GBS groups

Theorem (W.)

Let $G = \pi_1(\mathcal{G})$ *be a generalized Baumslag-Solitar group. Then the conjugacy problem of* G *is in* LOGSPACE.

Uniform Conjugacy in GBS groups

Input:

- a finite graph of groups $\mathcal{G}$ consisting of
  - (V, E),
  - $\alpha_t, \beta_t \in \mathbb{Z} \setminus \{0\}$ for $t \in E$ given in binary,
- two words $v, w \in \pi_1(\mathcal{G})$

Question: $v \sim w$ in $\pi_1(\mathcal{G})$.

Theorem (W.)

The uniform conjugacy problem for GBS groups is EXPSPACE*-hard.*

Proof.

The uniform reachability problem for symmetric Petri nets is EXPSPACE-complete (Mayr, Meyer, 1982). □

More General

Fundamental groups of finite graphs of groups with free abelian vertex and edge groups:

Conjecture

Word problem is in DET (i.e. NC^1-reducible to integer determinant, iterated matrix product, or matrix powering).

Theorem (Bogopolski, Martino, Ventura, 2010)

Conjugacy problem is undecidable in general.

Theorem (Diekert, Miasnikov, W., 2015)

Conjugacy problem is strongly generically in P *(except special case).*

Conclusion

- The word and conjugacy problem of generalized Baumslag-Solitar groups is in LOGSPACE.
- Conjecture: The uniform conjugacy problem for GBS groups is EXPSPACE-complete.
- Conjecture: The word problem of fundamental groups of finite graphs of groups with free abelian vertex and edge groups is in DET.

Conclusion

- The word and conjugacy problem of generalized Baumslag-Solitar groups is in LOGSPACE.
- Conjecture: The uniform conjugacy problem for GBS groups is EXPSPACE-complete.
- Conjecture: The word problem of fundamental groups of finite graphs of groups with free abelian vertex and edge groups is in DET.

Thank you!