Worst-Case Efficient Sorting with QuickMergesort

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San Diego, January 7, 2019

Comparison-based sorting: Quicksort, Heapsort, Mergesort

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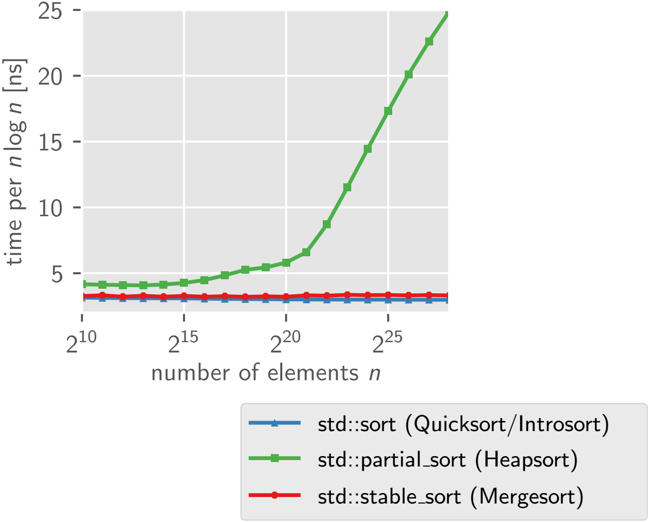
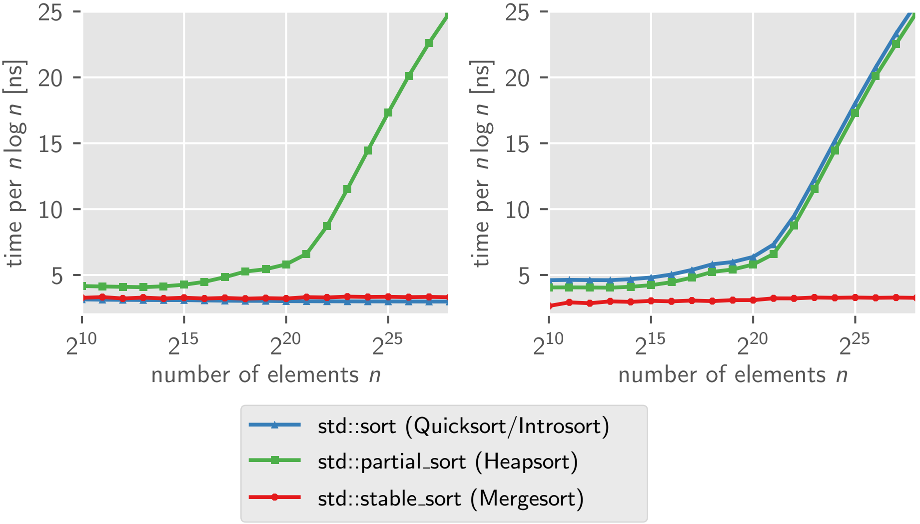


Running times (divided by $n \log n$) for sorting integers.

Left: random inputs.

Comparison-based sorting: Quicksort, Heapsort, Mergesort



Running times (divided by $n \log n$) for sorting integers.

Left: random inputs.

Right: random with large elements in the middle and end.

Quicksort, Heapsort, Mergesort

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Quicksort	✓	✓	✗
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  - rotation based merging (stable, but $\mathcal{O}(n \log^2 n)$)
  - use one half as buffer to sort the other half
    (In-situ Mergesort [Elmasry, Katajainen, Stenmark 2012], unstable)

Outline

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- QuickMergesort
- Our improvements and theoretical bounds
- Experiments

Quicksort

procedure $\textsc{Quicksort}(A[\ell, \ldots, r])$
  if $r > \ell$ **then**
    pivot $\leftarrow$ choosePivot($A[\ell, \ldots, r]$)
    cut $\leftarrow$ partition($A[\ell, \ldots, r]$, pivot)
    Quicksort($A[\ell, \ldots, \mathrm{cut} - 1]$)
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  end if
end procedure

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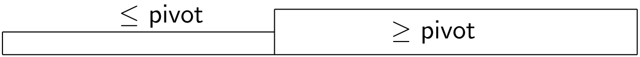

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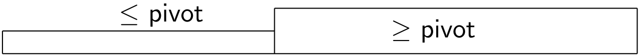

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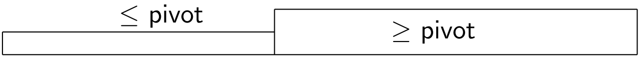
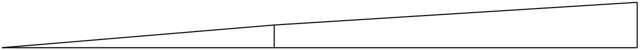


- After line 5:

- After line 6: both parts sorted recursively with Quicksort



QuickMergesort

```
procedure QUICKSORT(A[ℓ, ..., r])
    if r > ℓ then
        pivot ← choosePivot(A[ℓ, ..., r])
        cut ← partition(A[ℓ, ..., r], pivot)
        Quicksort(A[ℓ, ..., cut − 1])
        Quicksort(A[cut, ..., r])
    end if
end procedure
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procedure QUICKSORT($A[\ell, \dots, r]$)

    if $r > \ell$ **then**

        pivot $\leftarrow$ choosePivot($A[\ell, \dots, r]$)

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        Mergesort ~~Quicksort~~($A[\ell, \dots, \mathrm{cut} - 1]$)

        Quicksort($A[\mathrm{cut}, \dots, r]$)

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procedure QuickMergesort($A[\ell, \ldots, r]$)
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        QuickMergesort($A[\mathrm{cut}, \ldots, r]$)
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1: **procedure** QuickMergesort($A[\ell, \dots, r]$)
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3:         pivot $\leftarrow$ choosePivot($A[\ell, \dots, r]$)
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5:         Mergesort($A[\ell, \dots, \mathrm{cut} - 1]$)
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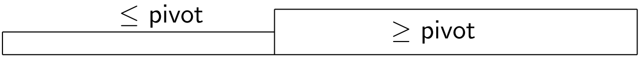

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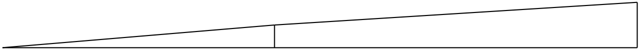
- After line 6: both parts sorted recursively with QuickMergesort



QuickMergesort

1. Partition according to some pivot element.
2. Sort one part with Mergesort.
3. Sort the the other part recursively with QuickMergesort.

7	11	4	5	6	10	9	2	3	1	0	8

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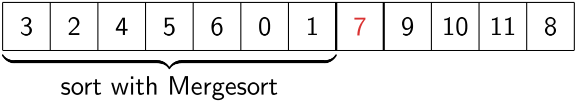

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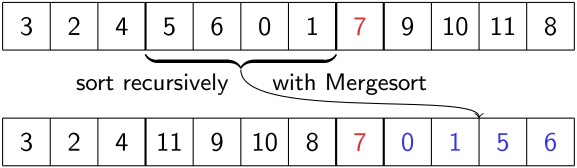

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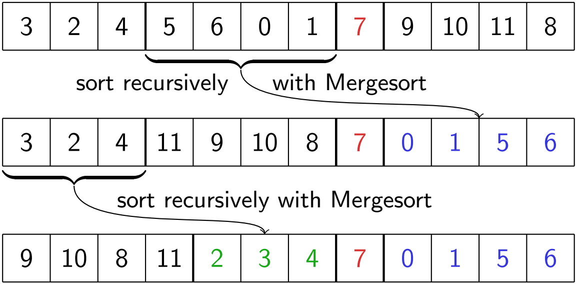

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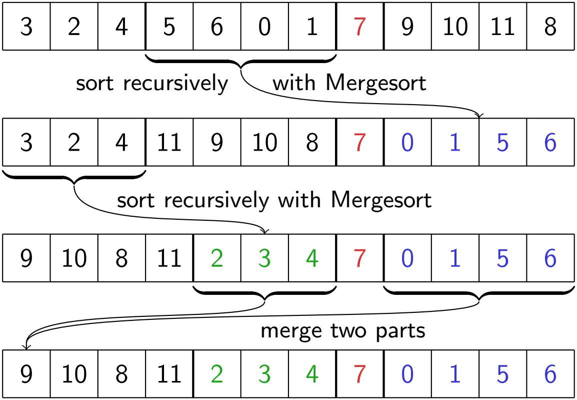

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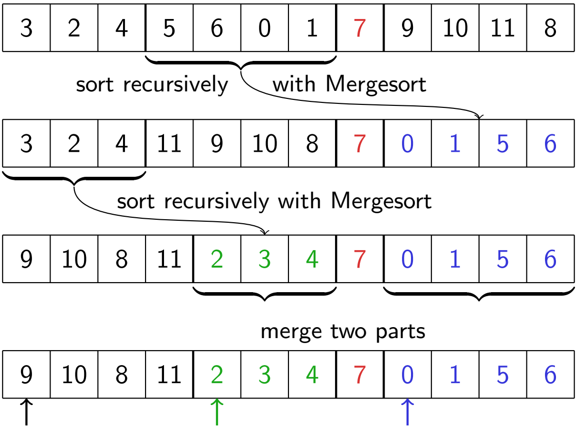

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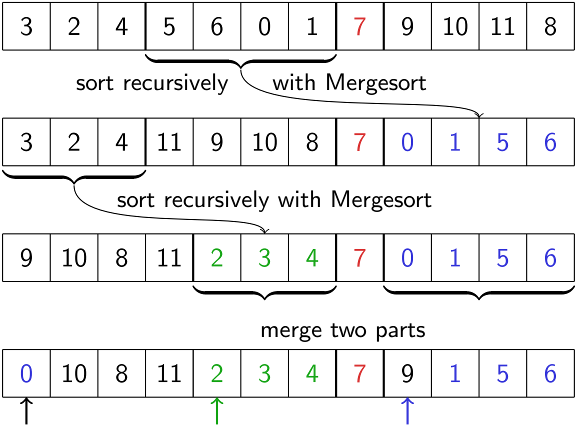

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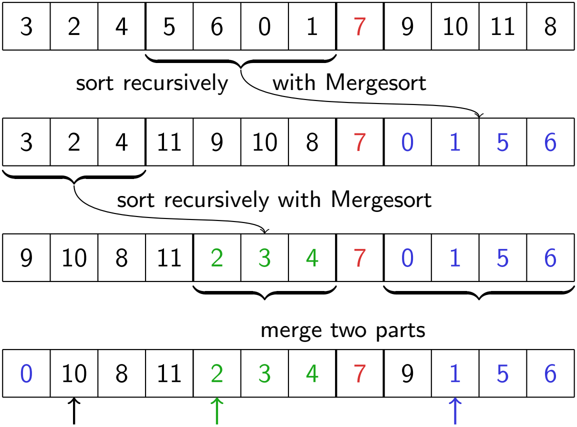

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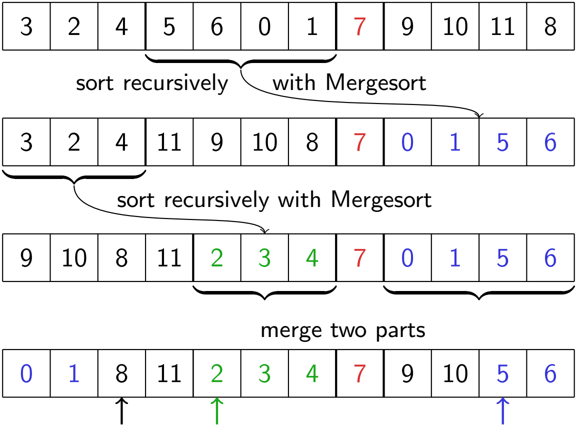

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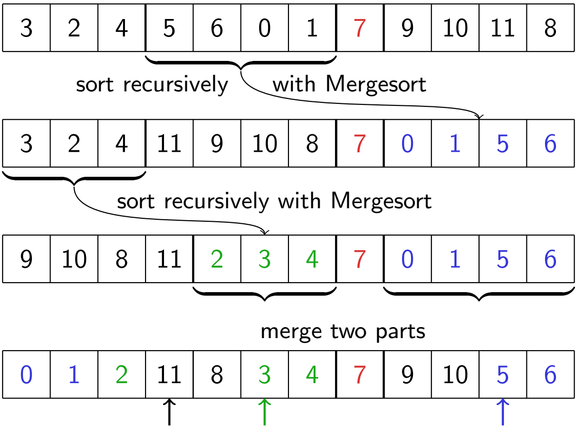

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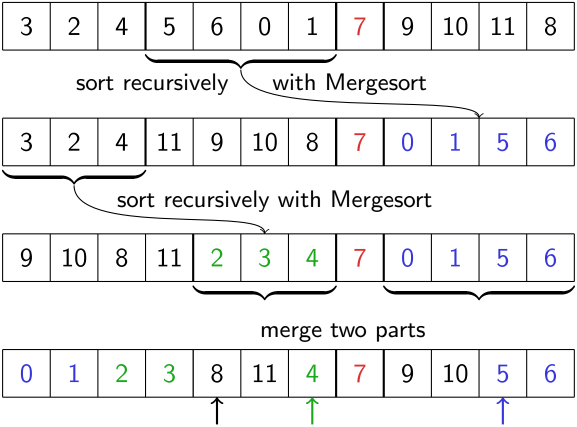

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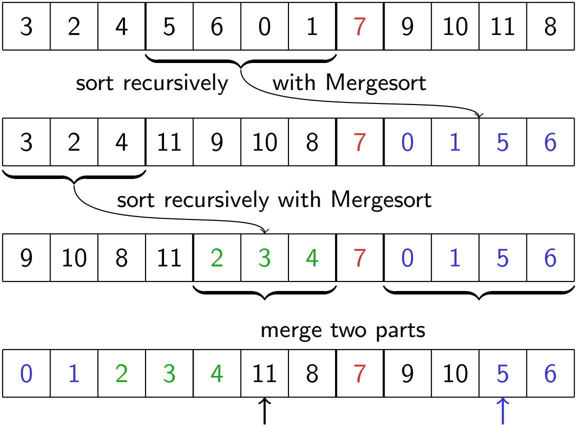

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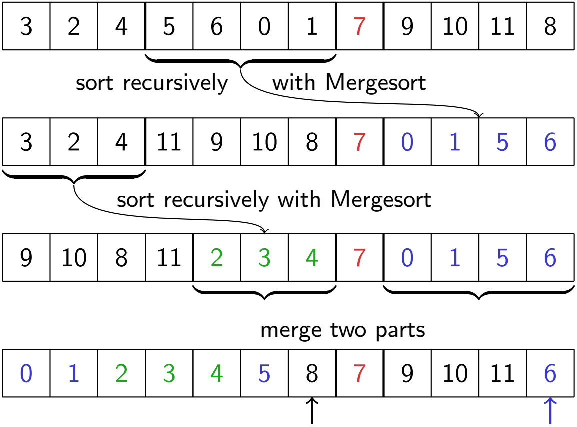

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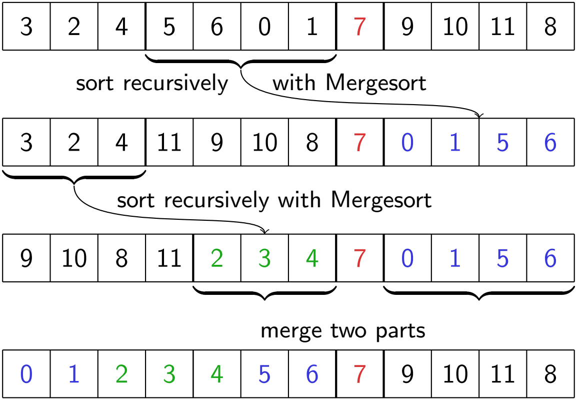

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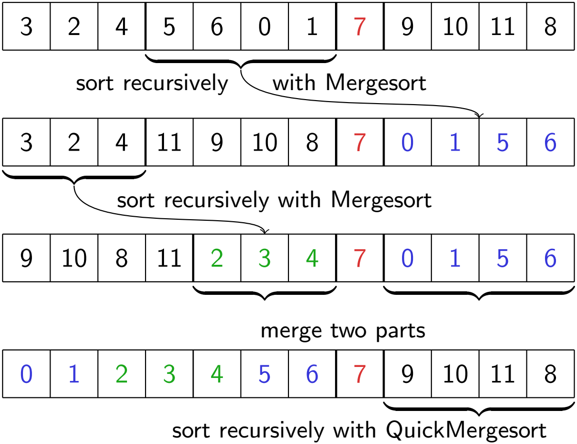

Worst case of QuickMergesort

Theorem (Edelkamp, W. 2014, Wild 2018)

QuickMergesort needs $\leq n \log n - 0.83n$ *(resp.* $n \log n - 1.24n$*)* $+o(n)$ *comparisons on average with median-of-three (resp. median of* $\sqrt{n}$*).*

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Need to find the median of n, $\frac{n}{2}$, $\frac{n}{4}$, $\ldots$ elements $\rightsquigarrow$ $40n$ comparisons

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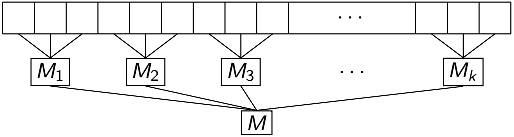
- form groups of 3 elements
- compute the median of each group
- compute the median of the $n/3$ medians



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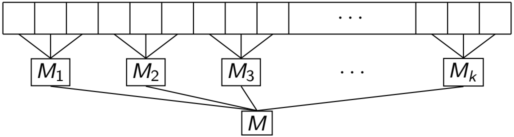

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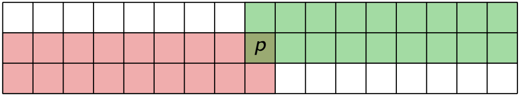

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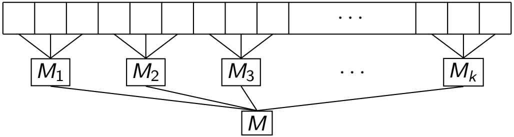

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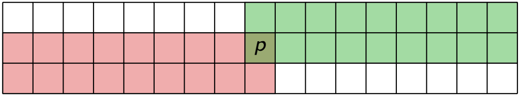
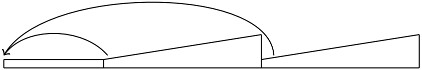


Theorem

Basic MoMQuickMergesort needs at most $n \log n + 13.8n + o(n)$ *comparisons.*

Merging with less buffer space (Reinhardt 1992)

- Step 1 (merge from the left): 

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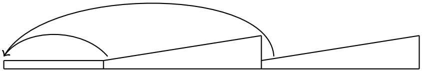
- Step 1 (merge from the left): 
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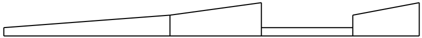
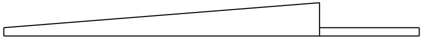


- Expected result: 

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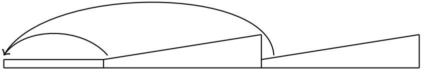
- Step 1 (merge from the left): 
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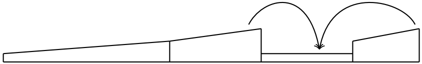
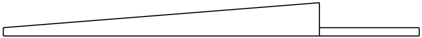
- Step 2 (merge from the right): 
- Result: 

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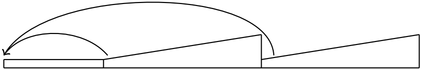
- Step 1 (merge from the left): 
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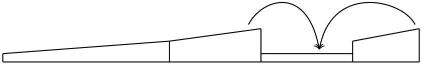
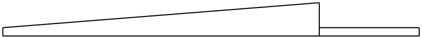
- Step 2 (merge from the right): 
- Result: 

Need a guarantee that one fifth are smaller/greater than the pivot:

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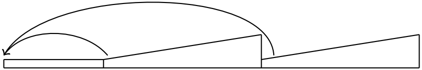
- Step 1 (merge from the left): 
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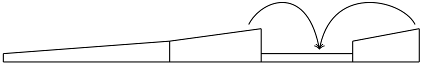
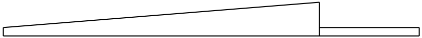
- Step 2 (merge from the right): 
- Result: 

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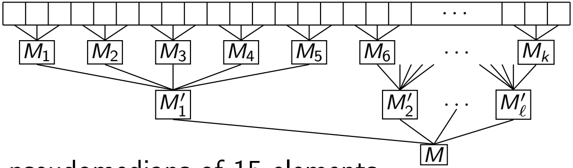

$\rightsquigarrow$ median of pseudomedians of 15 elements

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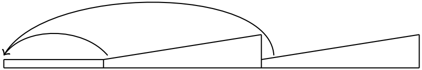
- Step 1 (merge from the left): 
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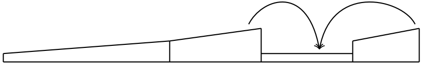
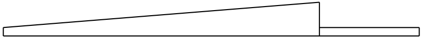
- Step 2 (merge from the right): 
- Result: 

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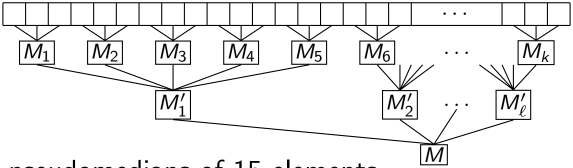


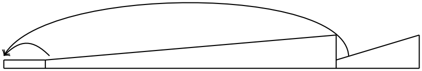
$\rightsquigarrow$ median of pseudomedians of 15 elements

Theorem

MoMQuickMergesort needs at most $n \log n + 4.57n + o(n)$ *comparisons.*

Unbalanced merging and undersampling

For merging sequences of different size a smaller buffer suffices:

- Step 1 (merge from the left): 

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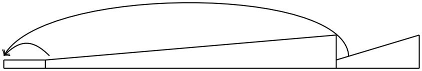
- Step 1 (merge from the left): 
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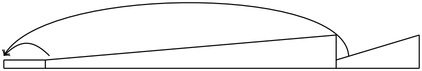
- Step 1 (merge from the left): 
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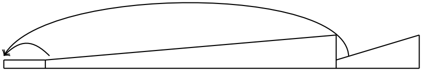
- Step 1 (merge from the left): 
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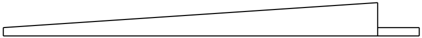
- Result: 

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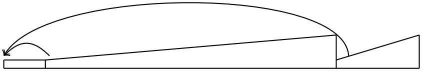

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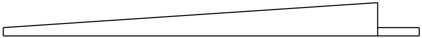

⇝ trade-off: effort to find good pivots vs. increased merging costs

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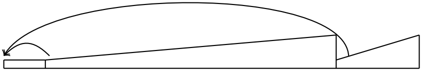

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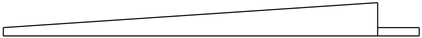

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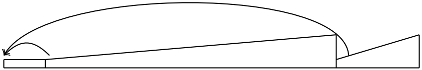

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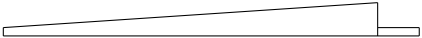

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Theorem

MoMQuickMergesort with undersampling factor $\theta = 2.2$ needs at most $n \log n + 1.59n + o(n)$ comparisons.

Experiments

- Experiments with random permutations of 32-bit integers (other data types in proceedings) in C++.

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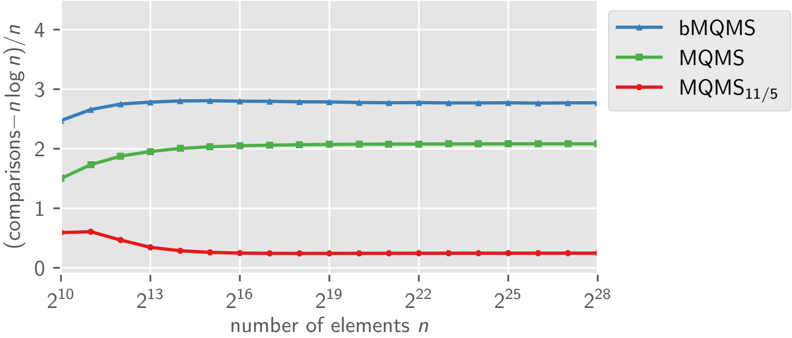
    - Minor details (e. g. random shuffle before Mergesort).

Counting comparisons

Algorithm	average case		worst case	
	exp.	theo.	exp.	theo.
bMQMS	2.772 ± 0.02			
MQMS	2.084 ± 0.001			
$\mathrm{MQMS}_{11/5}$	0.246 ± 0.01			



Number of comparisons (linear term) of MoMQuickMergesort variants

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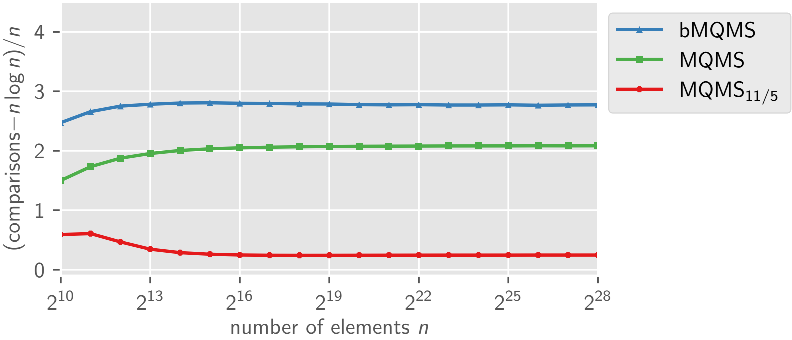

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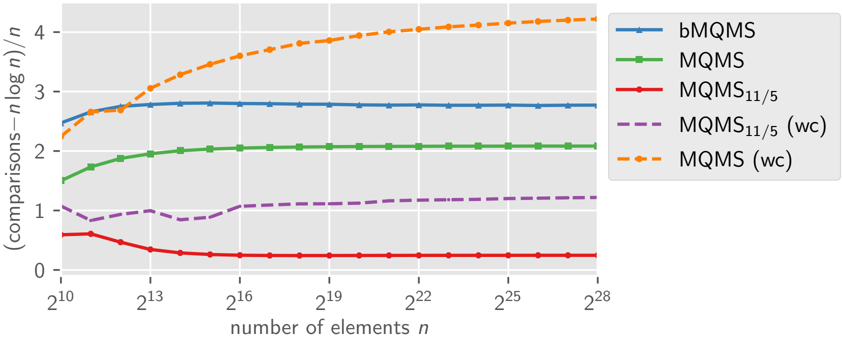
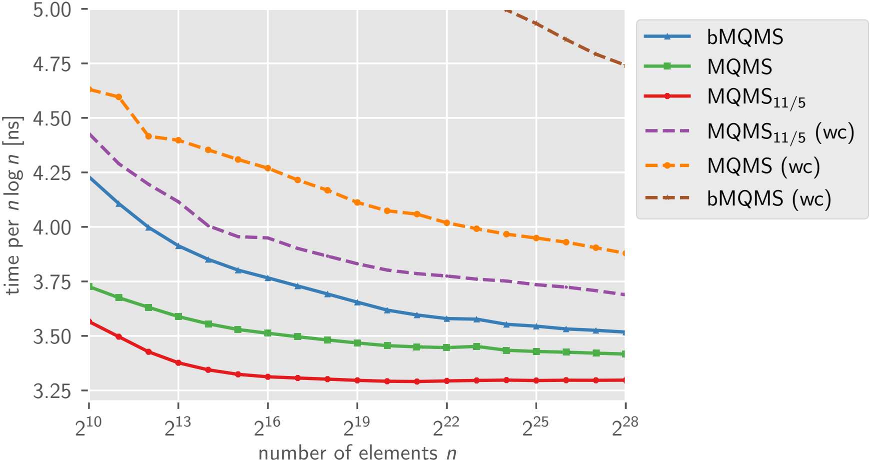
bMQMS	2.772 $\pm$ 0.02	–	13.05 $\pm$ 0.17	13.8
MQMS	2.084 $\pm$ 0.001	2.094	4.220 $\pm$ 0.007	4.57
$\mathrm{MQMS}_{11/5}$	0.246 $\pm$ 0.01	0.275	1.218 $\pm$ 0.011	1.59



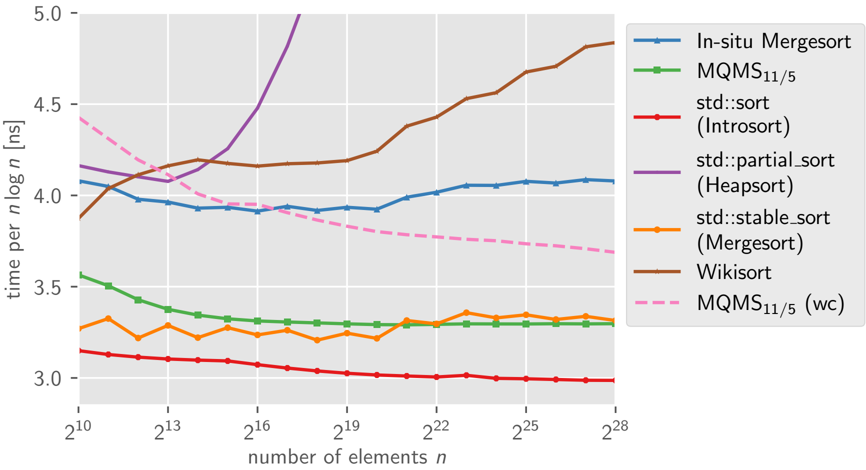
Number of comparisons (linear term) of MoMQuickMergesort variants and simulated worst cases.

Running times



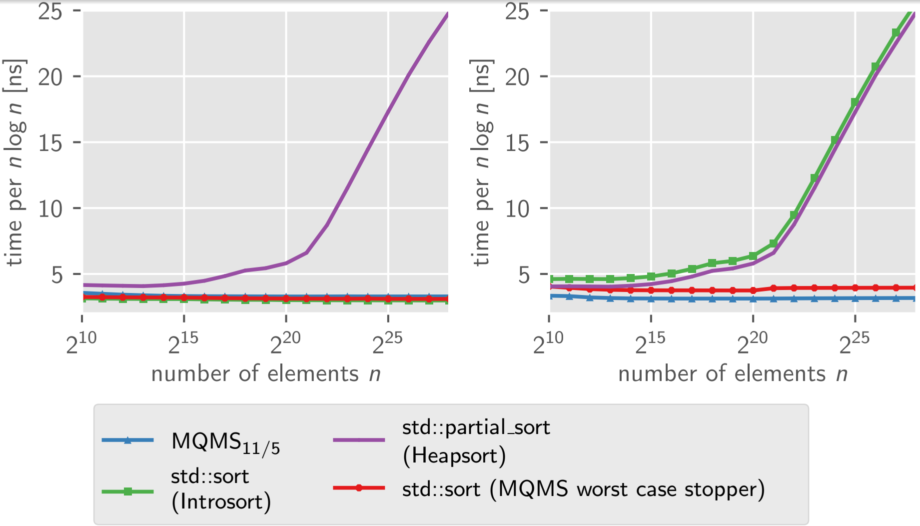
Running times of different MoMQuickMergesort variants and their simulated worst cases for random permutations of 32-bit integers.

Running times



Running times for random permutations of 32-bit integers.

Running times

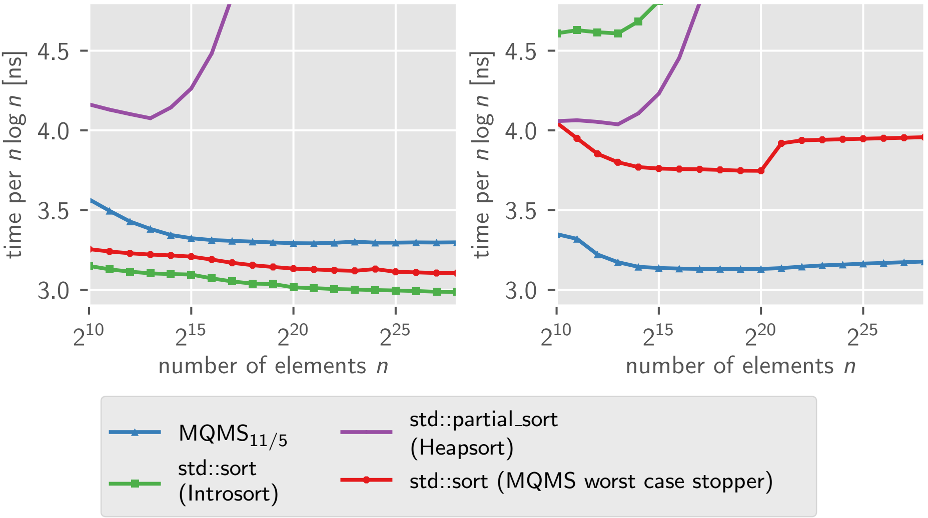


Running times for sorting integers.

Left: random inputs.

Right: Random with large elements in the middle and end.

Running times



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Algorithm	Fast on average	"in place"	$\mathcal{O}(n \log n)$ worst case
Quicksort	✓	✓	✗
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[1] Code available at `https://github.com/weissan/QuickXsort`

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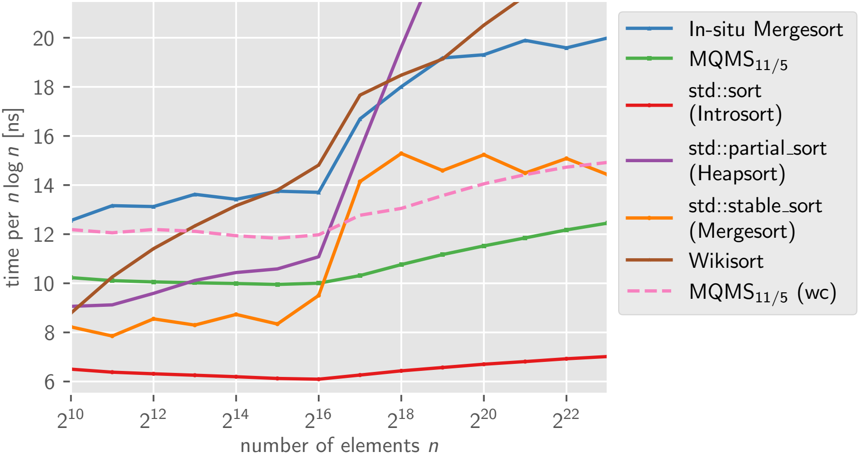
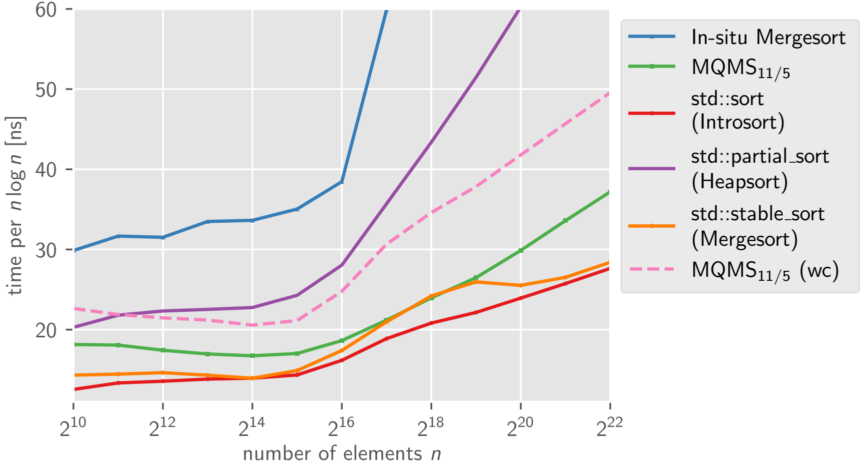
Experiments with larger records



Running times of MoMQuickMergesort (average and simulated worst case), hybrid QMS and other algorithms for random permutations 44-byte records with 4-byte keys.

Experiments sorting pointers



Running times of MoMQuickMergesort (average and simulated worst case), hybrid QMS and other algorithms for random permutations of pointers to records.

Experimental setup

- Experiments with random permutations of 32bit integers (other data types in proceedings).
- running time and comparison count
- ≥ 100 measurements for each data point
- Test environment:
  - Intel Core i5-2500K CPU (3.30GHz) with 16GB RAM
  - Ubuntu Linux 64bit version 14.04.4
  - g++ (4.8.4) compiler with flags `-O3 -march=native`